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# Bounds for some shape functions

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Abstract: In the finite element methods, some functions called *shape functions* are used. More precisely, given a triangulation $\mathcal{T}$ in $\mathbb{R}^2$, to each vertex and triangle of $\mathcal{T}$ one associates some shape functions. In this paper we determine bounds for some shape functions, with respect to the length of the adjacent edges. These bounds are useful in establishing properties for some interpolation operators (see [1]).

Key Words: Finite element, shape functions.

## 1 Preliminaries

Given a set of $V$ distinct points in $\mathbb{R}^2$, we construct a triangulation $\mathcal{T}$. Then, for each of the triangle of the given triangulation $\mathcal{T}$, some functions will be associated in the following way. Consider the triangle $T \in \mathcal{T}$ with the vertices $M_1(x_1, y_1), M_2(x_2, y_2), M_3(x_3, y_3)$ and the number $D_T = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$. Using $D_T$, we define the functions $A^T, B^T, C^T : \mathbb{R}^2 \to \mathbb{R}$,

$$
\begin{aligned}
A^T(x,y) &= \frac{(x_3-x_2)(y-y_3)-(y_3-y_2)(x-x_3)}{D_T} \\
&= \frac{(x_3-x_2)(y-y_2)-(y_3-y_2)(x-x_2)}{D_T}, \\
B^T(x,y) &= \frac{(x_1-x_3)(y-y_1)-(y_1-y_3)(x-x_1)}{D_T} \\
&= \frac{(x_1-x_3)(y-y_3)-(y_1-y_3)(x-x_3)}{D_T}, \\
C^T(x,y) &= \frac{(x_2-x_1)(y-y_2)-(y_2-y_1)(x-x_2)}{D_T} \\
&= \frac{(x_2-x_1)(y-y_1)-(y_2-y_1)(x-x_1)}{D_T}.
\end{aligned}
$$

The following proposition gives some immediate properties of these functions.

**Proposition 1.1** *The following statements are true.*

1. *If $M(x,y)$ is a point inside the triangle $T$, then $A^T(x,y) = \frac{area(MM_2M_3)}{area(M_1M_2M_3)}$;*

2. $A^T(x,y) \in [0,1]$, *for all* $(x,y) \in T$,

3. $A^T + B^T + C^T = 1$ *in* $T$;

4. *The restrictions of the function* $A^T, B^T, C^T$ *to the edges of the triangle* $T$ *are*

$$\begin{aligned} A^T|_{M_2M_3} &= 0,\ A^T|_{M_1M_3} = \frac{x-x_3}{x_1-x_3} = \frac{y-y_3}{y_1-y_3},\ A^T|_{M_1M_2} = \frac{x-x_2}{x_1-x_2} = \frac{y-y_2}{y_1-y_2}, \\ B^T|_{M_1M_3} &= 0,\ B^T|_{M_1M_2} = \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1},\ B^T|_{M_2M_3} = \frac{x-x_3}{x_2-x_3} = \frac{y-y_3}{y_2-y_3}, \\ C^T|_{M_1M_2} &= 0,\ C^T|_{M_2M_3} = \frac{x-x_2}{x_3-x_2} = \frac{y-y_2}{y_3-y_2},\ C^T|_{M_3M_1} = \frac{x-x_1}{x_3-x_1} = \frac{y-y_1}{y_3-y_1}. \end{aligned}$$

In the following definition, in order to simplify the writing we denote $A = A^T, B = B^T, C = C^T$.

**Definition 1.2** *The functions* $f^T_{M_1}, g^T_{M_1}, h^T_{M_1} : \mathbb{R}^2 \to \mathbb{R}$, *associated to the vertex* $M_1$ *of the triangle* $T$, *defined by*

$$\begin{aligned} f^T_{M_1} &= 2C^3 + 2B^3 - 3C^2 - 3B^2 + 1 - 4ABC, \\ g^T_{M_1} &= (x_3 - x_1)(C^3 - CB^2 - 2C^2) + (x_2 - x_1)(B^3 - BC^2 - 2B^2) + x - x_1, \\ h^T_{M_1} &= (y_3 - y_1)(C^3 - CB^2 - 2C^2) + (y_2 - y_1)(B^3 - BC^2 - 2B^2) + y - y_1, \end{aligned}$$

*are called* shape functions.
*Analogously, for the vertices* $M_2$ *and* $M_3$, *the functions are defined by circular permutations of the functions* $A, B, C$.

The following proposition summarizes some immediate properties of these functions.

**Proposition 1.3** *The following statements are true for* $i, j \in \{1,2,3\}$.

1. $f^T_{M_i}(M_j) = \delta_{ij},\ \frac{\partial f^T_{M_i}}{\partial x}(M_j) = 0,\ \frac{\partial f^T_{M_i}}{\partial y}(M_j) = 0$;

2. $g^T_{M_i}(M_j) = 0,\ \frac{\partial g^T_{M_i}}{\partial x}(M_j) = \delta ij,\ \frac{\partial g^T_{M_i}}{\partial y}(M_j) = 0$;

3. $h^T_{M_i}(M_j) = 0,\ \frac{\partial h^T_{M_i}}{\partial x}(M_j) = 0,\ \frac{\partial h^T_{M_i}}{\partial y}(M_j) = \delta_{ij}$;

4. *Along the edges of the triangle* $M_1M_2M_3$, *the functions* $f^T_{M_i}, g^T_{M_i}, h^T_{M_i}$ *depend only on the corresponding vertices;*

5. $f^T_{M_1} + f^T_{M_2} + f^T_{M_3} = 1$.

## 2 Bounds of the shape functions

We are interested in finding bounds for the shape functions $f^T_{M_i}, g^T_{M_i}, h^T_{M_i}$.

**Proposition 2.1** *Let* $T = M_1M_2M_3$, *with* $M_i(x_i, y_i)$, *be a triangle of* $\mathcal{T}$. *Then, for all* $(x,y) \in T$, *the functions* $f^T_{M_1}, g^T_{M_1}, h^T_{M_1}$ *have the following properties.*

*1.* $0 \le f^T_{M_1}(x,y) \le 1$;

*2.* $\left|g^T_{M_1}(x,y)\right| \le \max\left\{\frac{1}{4}|x_2-x_1|, \frac{1}{4}|x_3-x_1|\right\}$;

*3.* $\left|h^T_{M_1}(x,y)\right| \le \max\left\{\frac{1}{4}|y_2-y_1|, \frac{1}{4}|y_3-y_1|\right\}$.

*Proof.* In order to simplify the writing, we denote $f^T_{M_1} = f_{M_1},\ g^T_{M_1} = g_{M_1},\ h^T_{M_1} = h_{M_1}$.

1. Making the transform $\mu : \mathbb{R}^2 \to \mathbb{R}^2$ described by the equations

$$\begin{aligned} u &= u(x,y) = B + C, \\ v &= v(x,y) = B \cdot C,\ (x,y) \in \mathbb{R}^2, \end{aligned}$$

the triangle $T$ maps into the domain $W$ and the function $f_{M_1}(x,y) = f_{M_1}\left(\mu^{-1}(u,v)\right) = \xi(u,v)$ can be written as

$$\xi(u,v) = 2u^3 - 3u^2 - 2uv + 2v + 1.$$

The stationary point of the function $\xi$ is $(u,v) = (1,0)$, where $\xi$ takes the value 0. On the edges of the triangle $T$ we have

$$f_{M_1}|_{M_1M_2}(x,y) = 2\left(\frac{x-x_1}{x_2-x_1}\right)^3 - 3\left(\frac{x-x_1}{x_2-x_1}\right)^2 + 1 = 2t^3 - 3t^2 + 1,$$

with $t = (x-x_1)/(x_2-x_1),\ t \in [0,1]$. The function $\tau : [0,1] \to \mathbb{R}$,

$$\tau(t) = 2t^3 - 3t^2 + 1$$

satisfies the inequalities $0 \le \tau(t) \le 1$ for all $t \in [0,1]$ and thus

$$0 \le f_{M_1}|_{M_1M_2} \le 1.$$

Similarly,

$$0 \le f_{M_1}|_{M_1M_3} \le 1.$$

Since on the edge $M_2M_3$ we have $B + C = 1 - A = 1$, one obtains

$$f_{M_1}|_{M_2M_3} = \xi(1,v) = 0.$$

Therefore

$$0 \le f_{M_1}(x,y) \le 1 \text{ for all } (x,y) \in T.$$

2. Using the identity $x - x_1 - C(x_3 - x_1) - B(x_2 - x_1) = 0$, the function $g_{M_1}$ can be written as

$$g_{M_1} = (1 - B - C)\left(aC(1 - C + B) + bB(1 - B + C)\right),$$

where $a = x_3 - x_1,\ b = x_2 - x_1$. We consider two cases:
**Case 1** $x_2 = x_3$. Denoting

$$\varphi(B,C) = (1 - B - C)\left(C(1 - C + B) + B(1 - B + C)\right),$$

we have to find the extremes of $\varphi$ when $0 \le B \le 1,\ 0 \le C \le 1,\ B + C \le 1$. The stationary points of the function $\varphi$ are $(B,C) \in \left\{(1,0),(0,1),\left(\frac{1}{4},\frac{1}{4}\right)\right\}$, and at these points $\varphi$ takes the values $\varphi(1,0)=\varphi(0,1)=0,\ \varphi\left(\frac{1}{4},\frac{1}{4}\right)=\frac{1}{4}$. Therefore, in this case the extremes of $g_{M_1}$ are

$$\begin{aligned}\min_{(x,y)\in T} g_{M_1}(x,y) &= \min\left\{0,\frac{a}{4}\right\},\\ \max_{(x,y)\in T} g_{M_1}(x,y) &= \max\left\{0,\frac{a}{4}\right\},\end{aligned}$$

whence the conclusion $\left|g_{M_1}(x,y)\right| \le \left|\frac{a}{4}\right|$, for all $(x,y)\in T$.

**Case 2** $x_2 \ne x_3$. Making the transform $\omega:\mathbb{R}^2\to\mathbb{R}^2$ described by the functions

$$u=u(x,y)=C-B,\ v=v(x,y)=B+C,\ (u,v)\in\mathbb{R}^2,$$

the triangle $T$ maps into the domain $U$ and the function $g_{M_1}(x,y)=g_{M_1}\left(\omega^{-1}(u,v)\right)=\psi(u,v)$ becomes

$$\psi(u,v)=\frac{1}{2}(v-1)\left(b(1+u)(u-v)+a(u-1)(u+v)\right).$$

The values of $g_{M_1}$ on the edges are

$$\begin{aligned} g_{M_1}|_{M_1M_2} &= b\,B(1-B)^2,\\ g_{M_1}|_{M_1M_3} &= a\,C(1-C)^2,\\ g_{M_1}|_{M_2M_3} &= 0.\end{aligned}$$

Since the stationary points of the function $\sigma(\eta)=\eta(1-\eta)^2$ are $\eta=\frac{1}{3}$ and $\eta=1$ and the values of $\sigma$ at these points are $\sigma(\frac{1}{3})=\frac{4}{27},\ \sigma(1)=0$, we conclude that $g_{M_1}|_{M_1M_2}$ takes values between 0 and $\frac{4b}{27}$, while $g_{M_1}|_{M_1M_3}$ takes values between 0 and $\frac{4a}{27}$.
The stationary points of the function $\psi$ are

$$(u,v)\in\left\{(1,1),(-1,1),\left(\rho,\frac{b-a+2\rho(a+b)}{b-a}\right)\right\},$$

where $\rho$ is a root of the equation $3\,(b-a)\,z^2+4\,(b+a)\,z+b-a=0$, that is

$$\rho=-\frac{1}{3(b-a)}\left(2(a+b)\pm\sqrt{a^2+14ab+b^2}\right).$$

The values of the function $\psi(u,v)$ at the stationary points are

$$\begin{aligned}&\psi(1,1)=\psi(-1,1)=0,\\ &\psi\left(\rho,\frac{b-a+2\rho(a+b)}{b-a}\right)=\\ &=\frac{2(a+b)^2}{27(a-b)^4}\left((a+b)(a^2-34ab+b^2)\mp(a^2+14ab+b^2)^{\frac{3}{2}}\right).\end{aligned}$$

We have to decide which of the stationary points $(u,v)$ are situated inside the domain $U$. The stationary points $(1,1$ and $(-1,1)$ are situated on the border of $U$. Then we calculate that, at the stationary point $P_\rho\left(\rho,\frac{b-a+2\rho(a+b)}{b-a}\right)$, the functions $B$ and $C$ have the values

$$B = \frac{u+v}{2} = \frac{1}{2}\left(1+\rho\frac{3b+a}{b-a}\right),$$
$$C = \frac{v-u}{2} = \frac{1}{2}\left(1+\rho\frac{3a+b}{b-a}\right),$$

and with the notation $t = a/b$, these values can be written as

$$B = \frac{1}{2}\left(1+\rho\frac{3+t}{1-t}\right),$$
$$C = \frac{1}{2}\left(1+\rho\frac{3t+1}{1-t}\right)$$

The condition that the stationary point $P_\rho$ is situated inside the domain $U$ is equivalent to $B \in (0,1),\ C \in (0,1),\ B+C \le 1$. Conditions $B \in (0,1),\ C \in (0,1)$ are satisfied only by the root

$$\rho_2 = -\frac{1}{3(b-a)}\left(2(a+b)-\sqrt{a^2+14ab+b^2}\right)$$
$$= -\frac{1}{3(1-t)}\left(2(1+t)-\sqrt{1+14t+t^2}\right),$$

only in the case $a/b \ge 0$. Condition $B+C \le 1$ means $v \le 1$ or, equivalently,

$$1+2\rho_2\frac{a+b}{b-a} \le 1,$$
$$\frac{a+b}{3(b-a)^2}\left(2(a+b)-\sqrt{a^2+14ab+b^2}\right) \ge 0. \tag{1}$$

Thus, from the stationary points $P_1$ and $P_2$, only one of them ($P_2$) can be situated inside the domain $U$, if $a/b \in [0,\infty)\setminus\{1\}$. If $a=0$, then it is situated on an edge.
Subsequently, we consider now three subcases:
**Subcase 2a)** $a/b<0$. In this case the stationary points $P_1$ and $P_2$ are situated outside the domain $U$, so the extremes are taken on the edges. Therefore

$$\min_{(x,y)\in T} g_{M_1}(x,y) = \min\left\{0,\frac{4a}{27},\frac{4b}{27}\right\} = \min\left\{\frac{4a}{27},\frac{4b}{27}\right\}$$
$$\max_{(x,y)\in T} g_{M_1}(x,y) = \max\left\{\frac{4a}{27},\frac{4b}{27}\right\}$$

and finally we conclude that

$$\left|g_{M_1}(x,y)\right| \le \max\left\{\frac{4|a|}{27},\frac{4|b|}{27}\right\} \le \max\left\{\frac{|a|}{4},\frac{|b|}{4}\right\}. \tag{2}$$

for all $(x,y)\in T$,
**Subcase 2b )** $a<0,\ b<0$. In this case condition (1) becomes, with $t=a/b$,

$$(t+1)\left(2(1+t)-\sqrt{1+14t+t^2}\right) \le 0,$$

which, together with the condition $t \geq 0$ has no solution. Therefore, in this case the point $P_2$ is situated outside the domain $U$.
In conclusion, if $a < 0,\ b < 0$ the extreme values are taken on the edges and

$$|g_{M_1}(x,y)| \leq \max\left\{\frac{4|a|}{27}, \frac{4|b|}{27}\right\} \leq \max\left\{\frac{|a|}{4}, \frac{|b|}{4}\right\}. \tag{3}$$

**Subcase 2c )** $a > 0,\ b > 0$. With $t = a/b$, Condition (1) becomes

$$(t+1)\left(2(1+t) - \sqrt{1+14t+t^2}\right) \geq 0,$$

which, together with $t \geq 0$ gives $t \in [0,\infty) \setminus \{1\}$. We have to compare the values of $g_{M_1}$ on the edges with the values of $g_{M_1}$ at the stationary point $P_2$. The value of $g_{M_1}$ at the stationary point $P_2$ is

$$g_{M_1}(P_2) = \frac{2(a+b)^2}{27(a-b)^4}\left((a+b)(a^2-34ab+b^2) + (a^2+14ab+b^2)^{3/2}\right).$$

Condition $g_{M_1}(P_2) > \max\left\{\frac{4a}{27}, \frac{4b}{27}\right\}$ reduces to

$$\frac{2(t+1)^2}{27(t-1)^4}\left((t+1)(t^2-34t+1) + (t^2+14t+1)^{3/2}\right) \leq \frac{4}{27}t,$$

where

$$t = \begin{cases} a/b & \text{if } a < b, \\ b/a & \text{if } b < a \end{cases} = \frac{\min\{a,b\}}{\max\{a,b\}}$$

which is satisfied for all $t \geq 0,\ t \neq 1$.
Thus, $g(P_2) \geq \frac{4a}{27},\ g(P_2) \geq \frac{4b}{27}$ hold for all $\frac{a}{b} \geq 0,\ a \neq b$. Hence,

$$\max_{(x,y)\in T} g_{M_1}(x,y) = g_{M_1}(P_2).$$

and subsequently

$$\left|g_{M_1}(x,y)\right| \leq g_{M_1}(P_2)$$

for all $(x,y) \in T$. Let us remark that, for $a = 0,\ g_{M_1}(P_2) = \frac{4b}{27}$, while for $b = 0,\ g_{M_1}(P_2) = \frac{4a}{27}$.
Next we wish to find the smallest possible $\alpha$ such that

$$g_{M_1}(P_2) \leq \alpha a, \tag{4}$$
$$g_{M_1}(P_2) \leq \alpha b. \tag{5}$$

Denoting $t = \frac{\min\{a,b\}}{\max\{a,b\}}$, requirements (4) and (5) reduce to

$$\phi(t) = \frac{2(1+t)^2}{27(1-t)^4}\left((1+t)(t^2-34t+1) + (t^2+14t+1)^{3/2}\right) \leq \alpha t,\ t \in [0,1].$$

The function $\phi$ increases on $[0,1]$ and $\lim\limits_{t \nearrow 1} \frac{\phi(t)}{t} = \frac{1}{4}$, therefore the smallest possible $\alpha$ for which $\phi(t) \leq \alpha t$ is $\alpha = 1/4$, whence

$$g_{M_1}(P_2) \leq \max\left\{\frac{|a|}{4}, \frac{|b|}{4}\right\}.$$

Summarizing all the above arguments, we conclude that in all cases we have

$$\left|g_{M_1}(x,y)\right| \leq \max\left\{\frac{1}{4}|x_2 - x_1|, \frac{1}{4}|x_3 - x_1|\right\},$$

for all $(x,y) \in T$.

3. The proof is analogous with 2.

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