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On the Degree of Exactness of Some Positive Cubature Formulas on the Sphere

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Abstract: In [3] we studied some interpolatory cubature formulas associated to a fundamental system of $(n+1)^2$ ($n \in \mathbb{N}$ odd) points on the sphere, equidistributed on $n+1$ latitudinal circles. Being interpolatory, these formulas have the degree of exactness n, meaning that they are exact for spherical polynomials of degree $\leq n$. We gave also equivalent conditions under which the degree of exactness is $n+1$. In this paper we show that $n+1$ is the maximal degree of exactness attained by these formulas, under the assumption that the weights are positive.

1 Preliminaries

Let $\mathbb{S}^2 = \{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\|_2 = 1\}$ denote the unit sphere of the Euclidean space $\mathbb{R}^3$ and let

$$\begin{aligned} \Psi : [0,\pi] \times [0, 2\pi) &\rightarrow \mathbb{S}^2, \\ (\rho, \theta) &\mapsto (\sin\rho\cos\theta, \sin\rho\sin\theta, \cos\rho) \end{aligned}$$

be its parametrization in spherical coordinates (ρ, θ). The coordinate ρ of a point $\xi(\Psi(\rho,\theta)) \in \mathbb{S}^2$ is usually called the latitude of ξ.

We denote by Π_n the set of univariate polynomials of degree less than or equal to n, by $P_k,\ k = 0, 1, \ldots$, the Legendre polynomials of degree k on $[-1, 1]$, normalized by the condition $P_k(1) = 1$ and by V_n be the space of spherical polynomials of degree less than or equal to n. The dimension of V_n is $\dim V_n = (n+1)^2 = N$ and an orthogonal basis of V_n is given by

$$\left\{ Y_m^l(\theta,\rho) = P_m^{|l|}(\cos\rho)e^{il\theta},\ -m \leq l \leq m,\ 0 \leq m \leq n \right\}.$$

Here P_m^ν denotes the associated Legendre functions, defined by

$$P_m^\nu(t) = \left(\frac{(k-\nu)!}{(k+\nu)!}\right)^{1/2} (1-t^2)^{\nu/2} \frac{d^\nu}{dt^\nu} P_m(t),\ \nu = 0, \ldots, m,\ t \in [-1, 1]$$

and for given functions $f, g : \mathbb{S}^2 \to \mathbb{C}$, the inner product is taken as

$$\langle f, g \rangle = \int_{\mathbb{S}^2} f(\xi)\overline{g(\xi)}\, d\omega(\xi),$$

where $d\omega(\xi)$ stands for the surface element of the sphere.

The reproducing kernel of the space V_n is defined by

$$K_n(\xi,\eta) = \sum_{k=0}^{n} \frac{2k+1}{4\pi} P_k(\xi\cdot\eta) = k_n(\xi\cdot\eta), \quad \xi,\eta\in\mathbb{S}^2.$$

For given n we consider a set of points $\{\xi_i\}_{i=1,\dots,N} \subset \mathbb{S}^2$ and the polynomial functions $\varphi_i^n : \mathbb{S}^2 \to \mathbb{C},\ i=1,\dots,N$, defined by

$$\varphi_i^n(\circ) = K_n(\xi_i,\circ) = \sum_{k=0}^{n} \frac{2k+1}{4\pi} P_k(\xi_i\cdot\circ),\ i=1,\dots,N.$$

These polynomials are called *scaling functions.* A set of points $\{\xi_i\}_{i=1,\dots,N}$ for which the scaling functions $\{\varphi_i^n\}_{i=1,\dots,N}$ constitute a basis for V_n is called a *fundamental system* for V_n.

Recently, Laín Fernández proved the following result.

Proposition 1.1 *[1, 2] Let $n\in\mathbb{N}$ be an odd number, $\alpha\in(0,2)$ and let $0<\rho_1<\rho_2<\dots<\rho_{\frac{n+1}{2}}<\pi/2,\ \rho_{n+2-j}=\pi-\rho_j,\quad j=1,\dots,(n+1)/2$, denote a system of symmetric latitudes. Then the set of points $\mathcal{S}_n(\alpha)=\left\{\xi_{j,k}(\Psi(\rho_j,\theta_k^j)),\ j,k=1,\dots,n+1\right\}$, where*

$$\theta_k^j = \begin{cases} \frac{2k\pi}{n+1}, & \text{if } j \text{ is odd},\\ \frac{2(k-1)+\alpha}{n+1}\pi, & \text{if } j \text{ is even}, \end{cases}$$

constitutes a fundamental system for V_n.

Let us mention that for $\alpha=0$ or $\alpha=2$, the set $\mathcal{S}_n(\alpha)$ does not constitute a fundamental system of points and not many fundamental systems of points are known in the present.

In [3] we studied the interpolatory cubature formula

$$\int_{\mathbb{S}^2} F(\xi)\,d\omega(\xi) \approx \sum_{j=1}^{n+1}\sum_{k=1}^{n+1} w_{j,k}^n F(\xi_{j,k}), \tag{1}$$

for odd n, with the nodes $\xi_{j,k}$ in $\mathcal{S}_n(\alpha)$. For the weights of this formula we have proved the equalities $w_{j,k}^n = w_j^n = w_j$ for $j,k\in\{1,\dots,n+1\}$ and the fact that the weights $w_j,\ j=1,\dots,n+1$, take $q=\frac{n+1}{2}$ distinct values denoted $\frac{2\pi}{n+1}a_j,\ j=1,\dots,q$. As expected, we obtained that $w_{n+2-j}=w_j$ for $j=1,\dots,n+1$, meaning that the weights corresponding to symmetric latitudes are equal.

Being interpolatory, this cubature formula has the degree of exactness n. In [3], Theorem 7, we proved that the degree of exactness can be $n+1$ if and only if $\alpha=1$ and

$$\sum_{j=1}^{n+1} w_j P_{n+1}(\cos\rho_j) = 0, \tag{2}$$

under the assumption that the weights are positive. A possible case is when ρ_j are taken as the roots of the Legendre polynomial P_{n+1}. In the sequel we intend to study whether the degree of exactness can be greater than $n+1$.

2 The degree of exactness $n+2$ cannot be reached

We want to study whether the cubature formula (1) can have positive weights and at the same time can be exact for all spherical polynomials in V_{n+2}. So we need to suppose $\alpha=1$ and that condition (2) is fulfilled. Therefore

$$\theta_k^j = \frac{\beta_j + 2k\pi}{n+1}, \text{ with } \beta_j = \begin{cases} 0, & \text{if } j \text{ is odd}, \\ -\pi, & \text{if } j \text{ is even}. \end{cases}$$

In addition, we impose exactness for the spherical polynomials

$$\left\{ Y_{n+2}^l(\theta,\rho),\ l=-n-2,\ldots,n+2 \right\},$$

knowing that this set constitutes a basis of the space $\mathrm{Harm}V_{n+1} = V_{n+2} \ominus V_{n+1}$. On the one hand, evaluating the integral in (1) for these spherical polynomials we get

$$\int_{\mathbb{S}^2} P_{n+2}^{|l|}(\cos\rho) e^{il\theta} d\omega(\xi) = \int_0^\pi P_{n+2}^{|l|}(\cos\rho)\sin\rho\, d\rho \int_0^{2\pi} e^{il\theta} d\theta.$$

But

$$\int_0^{2\pi} e^{il\theta} d\theta = \begin{cases} 2\pi, & \text{for } l=0, \\ 0, & \text{otherwise}. \end{cases}$$

On the other hand, evaluating the sums in (1) for these spherical polynomials we get

$$\begin{aligned} \sum_{j=1}^{n+1}\sum_{k=1}^{n+1} w_j P_{n+2}^{|l|}(\cos\rho_j) e^{il\theta_k^j} &= \sum_{j=1}^{n+1} w_j P_{n+2}^{|l|}(\cos\rho_j) \sum_{k=1}^{n+1} e^{il\frac{\beta_j+2k\pi}{n+1}} \\ &= \sum_{j=1}^{n+1} w_j P_{n+2}^{|l|}(\cos\rho_j) e^{il\frac{\beta_j}{n+1}} \sum_{k=1}^{n+1} e^{il\frac{2k\pi}{n+1}}. \end{aligned}$$

The last sum is zero if $l \notin (n+1)\mathbb{Z}$ and is $n+1$ if $l \in (n+1)\mathbb{Z}$.
With the above remarks, the quadrature formula (1) is exact for Y_{n+2}^l with $l \neq 0$ and $|l| \neq n+1$. It remains to impose that (1) is exact for Y_{n+2}^0 and $Y_{n+2}^{\pm(n+1)}$.

- In order to be exact for $l=0$, we should have

$$2\pi \int_0^\pi P_{n+2}(\cos\rho)\sin\rho\, d\rho = (n+1)\sum_{j=1}^{n+1} w_j \sum_{k=1}^{n+1} P_{n+2}(\cos\rho_j),$$

which yields

$$(0=) \int_{-1}^1 P_{n+2}(x)\, dx = \frac{n+1}{2\pi} \sum_{j=1}^{n+1} w_j P_{n+2}(\cos\rho_j).$$

With the notations $\cos\rho_j = r_j$, $a_j = \frac{n+1}{2\pi} w_j$, we get

$$\sum_{j=1}^{n+1} a_j P_{n+2}(r_j) = 0. \tag{3}$$

This last condition, added to the conditions of exactness for the polynomials Y_j^0 for $j = 0, \ldots, n+1$, conditions which can be written as

$$\sum_{j=1}^{n+1} a_j P_k(r_j) = 0, \text{ for } k = 0, 1, \ldots, n+1, \tag{4}$$

means that the one-dimensional quadrature formula

$$\int_{-1}^{1} f(x)dx \approx \sum_{j=1}^{n+1} a_j f(r_j) \tag{5}$$

should be exact for the Legendre polynomials $P_0, \ldots, P_{n+2}$ and therefore for all univariate polynomials in Π_{n+2}. Actually, condition (3) brings the additional condition that formula (5) is exact for the monomial x^{n+2}. This fact is true for odd n, due to the symmetry of the weights and of the latitudinal circles. In conclusion formula (1) is exact for the spherical polynomial Y_{n+2}^0.

- In order to be exact for the spherical polynomials $Y_{n+2}^{\pm(n+1)}$, we should have

$$\sum_{j=1}^{n+1} a_j P_{n+2}^{n+1}(\cos\rho_j)e^{i\beta_j} = 0,$$

and further, replacing P_{n+2}^{n+1} and β_j, this condition becomes

$$\sum_{j=1}^{n+1} (-1)^j a_j (1 - r_j^2)^q r_j = 0, \tag{6}$$

again with $r_j = \cos\rho_j$ for $j = 1, \ldots, n+1$ and $q = (n+1)/2$. Moreover, it can be rewritten as

$$\sum_{j=1,\ j \text{ even}}^{n+1} a_j (1 - r_j^2)^q r_j - \sum_{j=1,\ j \text{ odd}}^{n+1} a_j (1 - r_j^2)^q r_j = 0. \tag{7}$$

On the other hand, using the fact that (5) is exact for the odd polynomial $(1 - x^2)^q x$, we obtain that

$$\sum_{j=1}^{n+1} a_j r_j (1 - r_j^2)^q = 0,$$

whence

$$-\sum_{j=1,\ j \text{ odd}}^{n+1} a_j (1 - r_j^2)^q r_j = \sum_{j=1,\ j \text{ even}}^{n+1} a_j (1 - r_j^2)^q r_j.$$

Thus (7) becomes

$$\sum_{j=1,\ j \text{ even}}^{n+1} a_j (1 - r_j^2)^q r_j = 0.$$

This equality cannot be true under the assumption that the weights are positive, therefore formula (1) cannot be exact for the spherical polynomials $Y_{n+2}^{\pm(n+1)} \in V_{n+2}$.

In conclusion the maximal degree attained by the interpolatory positive cubature formula (1) is $n+1$.

Remark 2.1 *An improvement of the degree of exactness can be however achieved (see [4]) by taking more than* $n+1$ *equidistributed points on each latitudinal circle and arbitrary deviations* β_j*. Thus we do not use fundamental systems of points anymore and therefore the corresponding cubature formulas are not interpolatory. In this way we could obtain positive cubature formulas with* $(n+1)(2n+1)$ *points and degree of exactness* $2n+1$*.*

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