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Finite element operators for scattered data

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Abstract. Given a set of scattered data, we construct a triangulation and a bivariate finite element operator associated to this triangulation. This operator has the degree of exactness one and is local, in the sense that the information around the interpolation nodes are taken from a small neighborhood of that node. Then we determine a bound for the rest of the interpolation formula and make a comparison with another operator with degree of exactness one, constructed as a combined finite element operator.

1 Preliminaries

Given a set of V distinct points in $\mathbb{R}^2$, we construct a triangulation $\mathcal{T}$ and denote $\Delta \subseteq \mathbb{R}^2$ the set covered by the triangles of $\mathcal{T}$. Then, for each of the triangle of the given triangulation $\mathcal{T}$, some functions will be associated in the following way. Consider the triangle $T \in \mathcal{T}$ with the vertices $M_1(x_1, y_1)$, $M_2(x_2, y_2)$, $M_3(x_3, y_3)$ and the number $D_T = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$. Using D_T, we define the functions

$A^T, B^T, C^T : \mathbb{R}^2 \to \mathbb{R}$,

$$\begin{aligned}
A^T(x,y) &= \frac{(x_3-x_2)(y-y_3)-(y_3-y_2)(x-x_3)}{D_T} \\
&= \frac{(x_3-x_2)(y-y_2)-(y_3-y_2)(x-x_2)}{D_T}, \\
B^T(x,y) &= \frac{(x_1-x_3)(y-y_1)-(y_1-y_3)(x-x_1)}{D_T} \\
&= \frac{(x_1-x_3)(y-y_3)-(y_1-y_3)(x-x_3)}{D_T}, \\
C^T(x,y) &= \frac{(x_2-x_1)(y-y_2)-(y_2-y_1)(x-x_2)}{D_T} \\
&= \frac{(x_2-x_1)(y-y_1)-(y_2-y_1)(x-x_1)}{D_T}.
\end{aligned}$$

In the following definition, in order to simplify the writing we denote $A = A^T, B = B^T, C = C^T$.

Definition 1.1 *The functions* $f^T_{M_1}, g^T_{M_1}, h^T_{M_1} : \mathbb{R}^2 \to \mathbb{R}$, *associated to the vertex* M_1 *of the triangle* T, *defined by*

$$\begin{aligned}
f^T_{M_1} &= 2C^3 + 2B^3 - 3C^2 - 3B^2 + 1 + \alpha ABC, \\
g^T_{M_1} &= (x_3-x_1)(C^3 - CB^2 - 2C^2) + (x_2-x_1)(B^3 - BC^2 - 2B^2) \\
&\quad + x - x_1 + \beta ABC, \\
h^T_{M_1} &= (y_3-y_1)(C^3 - CB^2 - 2C^2) + (y_2-y_1)(B^3 - BC^2 - 2B^2) \\
&\quad + y - y_1 + \gamma ABC,
\end{aligned}$$

are called shape functions.
Analogously, for the vertices M_2 *and* M_3, *the functions are defined by circular permutations of the functions* A, B, C.

The following proposition summarizes some immediate properties of these functions.

Proposition 1.2 *The following statements are true for* $i, j \in \{1,2,3\}$

1. $f^T_{M_i}(M_j) = \delta_{ij}, \ \frac{\partial f^T_{M_i}}{\partial x}(M_j) = 0, \ \frac{\partial f^T_{M_i}}{\partial y}(M_j) = 0,$
2. $g^T_{M_i}(M_j) = 0, \ \frac{\partial g^T_{M_i}}{\partial x}(M_j) = \delta ij, \ \frac{\partial g^T_{M_i}}{\partial y}(M_j) = 0,$
3. $h^T_{M_i}(M_j) = 0, \ \frac{\partial h^T_{M_i}}{\partial x}(M_j) = 0, \ \frac{\partial h^T_{M_i}}{\partial y}(M_j) = \delta_{ij},$

4. *Along the edges of the triangle $M_1M_2M_3$, the functions $f^T_{M_i}, g^T_{M_i}, h^T_{M_i}$ depend only on the corresponding vertices,*

5. $f^T_{M_1} + f^T_{M_2} + f^T_{M_3} = 1$ *if and only if* $\alpha = -4$.

2 A local interpolant

Now, to a given function $\varphi : \mathbb{R}^2 \to \mathbb{R}$, $\varphi \in C^1(\Delta)$, we associate the polynomial function

$$(\mathcal{P}_T\varphi)(x,y) = \sum_{i=1}^{3} \varphi(M_i) f^T_{M_i}(x,y) + \frac{\partial\varphi}{\partial x}(M_i) g^T_{M_i}(x,y) + \frac{\partial\varphi}{\partial y}(M_i) h^T_{M_i}(x,y),$$

for $(x,y) \in T$. This polynomial function satisfies the interpolating conditions

$$(\mathcal{P}_T\varphi)(M_i) = \varphi(M_i),\ \frac{\partial\mathcal{P}_T}{\partial x}(M_i) = \frac{\partial\varphi}{\partial x}(M_i),\ \frac{\partial\mathcal{P}_T}{\partial y}(M_i) = \frac{\partial\varphi}{\partial y}(M_i),$$

for $i \in \{1,2,3\}$.
We define now the "global" interpolating piecewise polynomial function associated to the function φ as

$$(\mathcal{P}\varphi)(x,y) = \sum_{i=1}^{V} \varphi(M_i) F_i(x,y) + \frac{\partial\varphi}{\partial x}(M_i) G_i(x,y) + \frac{\partial\varphi}{\partial y}(M_i) H_i(x,y), \tag{2.1}$$

for $(x,y) \in \Delta$, where

$$F_i(x,y) = \begin{cases} f^T_{M_i}(x,y), & \text{if } (x,y) \text{ is inside or on the edges of the} \\ & \quad \text{triangle } T \in \mathcal{T} \text{ which has } M_i \text{ as vertex,} \\ 0, & \text{on the triangles which do not contain } M_i \end{cases}$$

and analogous definitions for the functions G_i and H_i. Figures 1,2 and 3 show the graphs of some functions F_i, G_i, H_i.
The functions F_i, G_i, H_i are continuous on Δ due to the property 4 of Proposition 1.2, so the piecewise polynomial function $\mathcal{P}$ is continuous on Δ.

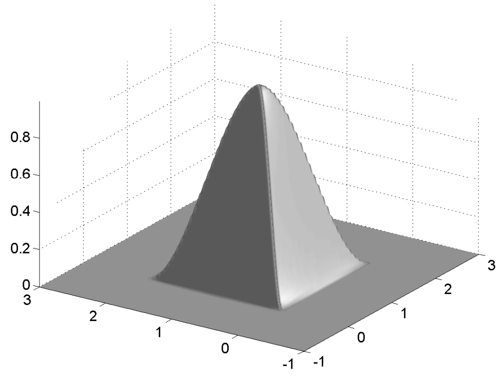

Figure 1: A function F_i.

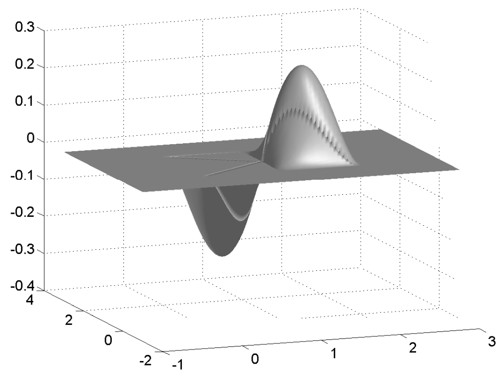

Figure 2: A function G_i.

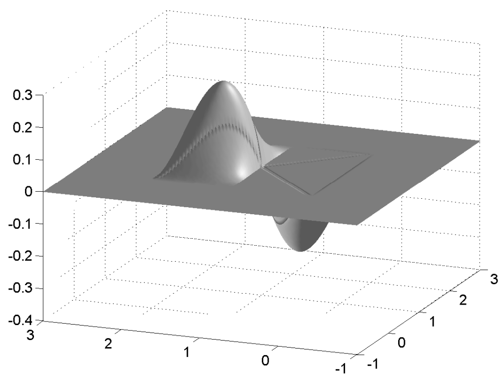

Figure 3: A function H_i.

It is also immediate that $\mathcal{P}\varphi$ satisfies the interpolating conditions

$$(\mathcal{P}\varphi)(M_i) = \varphi(M_i),\ \frac{\partial(\mathcal{P}\varphi)}{\partial x}(M_i) = \frac{\partial\varphi}{\partial x}(M_i),\ \frac{\partial(\mathcal{P}\varphi)}{\partial y}(M_i) = \frac{\partial\varphi}{\partial y}(M_i), \tag{2.2}$$

for $i = 1, \ldots, V$.
Other properties of the interpolating operator $\mathcal{P} : C^1(\Delta) \to C^1(\Delta)$, defined in (2.1), are given in the following proposition.

Proposition 2.1 *The operator $\mathcal{P}$ has the following reproducing properties*

1. $\mathcal{P}\varphi = \varphi$ *for* $\varphi =$ *constant if and only if* $\alpha = -4$,
2. $\mathcal{P}\varphi = \varphi$ *for* $\varphi(x, y) = x$ *if and only if* $\beta = 0$,
3. $\mathcal{P}\varphi = \varphi$ *for* $\varphi(x, y) = y$ *if and only if* $\gamma = 0$.

Proof. A calculation shows that

$$\sum_{i=1}^{V} F_i(x, y) = 1, \tag{2.3}$$

$$\sum_{i=1}^{V} x_i F_i(x, y) + G_i(x, y) = x. \tag{2.4}$$

$$\sum_{i=1}^{V} y_i F_i(x, y) + H_i(x, y) = y. \tag{2.5}$$

if and only if $\alpha = -4$ and $\beta = \gamma = 0$. Thus, for $\alpha = -4$ and $\beta = \gamma = 0$, the operator reproduces the polynomials of degree ≤ 1 in variables x and y. In this particular case, the bounds of the shape functions $f^T_{M_1}, g^T_{M_1}, h^T_{M_1}$ were established in [2], where the following result was proved.

Proposition 2.2 *Let $T = M_1M_2M_3$, $M_i(x_i, y_i)$, $i = 1, 2, 3$, be a triangle of $\mathcal{T}$ Then, for all $(x, y) \in T$, the functions $f^T_{M_1}, g^T_{M_1}, h^T_{M_1}$ have the following properties.*

1. $0 \leq f^T_{M_1}(x, y) \leq 1$,
2. $\left|g^T_{M_1}(x, y)\right| \leq \max\left\{\frac{1}{4}|x_2 - x_1|, \frac{1}{4}|x_3 - x_1|\right\}$,
3. $\left|h^T_{M_1}(x, y)\right| \leq \max\left\{\frac{1}{4}|y_2 - y_1|, \frac{1}{4}|y_3 - y_1|\right\}$.

Regarding the rest $\mathcal{R}\varphi$ of the interpolation formula

$$\varphi = \mathcal{P}\varphi + \mathcal{R}\varphi, \tag{2.6}$$

we have the following theorem.

Theorem 2.3 *Let $\varphi \in C^2(\Delta)$. Then we have*

$$\|\mathcal{R}\varphi\|_\infty = \sup_{(x,y)\in\,\Delta} |\varphi(x,y) - (\mathcal{P}\varphi)(x,y)| \leq 3(1+\sqrt{2}/2)K_2 L_{max}^2, \tag{2.7}$$

where

$$K_2 = \sup_{(x,y)\in\, int\,\Delta} \left\{ \left|\frac{\partial^2\varphi}{\partial x^2}(x,y)\right|, \left|\frac{\partial^2\varphi}{\partial y^2}(x,y)\right|, \left|\frac{\partial^2\varphi}{\partial x\partial y}(x,y)\right| \right\}$$

and L_{max} is the length of the greatest edge of the triangles of $\mathcal{T}$.

Proof. Before starting the proof we remark that, for given (x,y), in the sum (2.1), only at most three terms[1] are nonzero, namely those which correspond to the vertices of the triangle T for which $(x,y) \in T$.
So, let $(x,y) \in int\,\Delta$. Writing some Taylor formulas around the point (x,y) we get, for all $1 \leq i \leq V$,

$$\begin{aligned}
\varphi(M_i) &= \varphi(x,y) + (x_i - x)\frac{\partial\varphi}{\partial x}(x,y) + (y_i - y)\frac{\partial\varphi}{\partial y}(x,y) + R_i(x,y),\\
\frac{\partial\varphi}{\partial x}(M_i) &= \frac{\partial\varphi}{\partial x}(x,y) + S_i(x,y),\\
\frac{\partial\varphi}{\partial y}(M_i) &= \frac{\partial\varphi}{\partial y}(x,y) + T_i(x,y),
\end{aligned}$$

[1]In the case when (x,y) is situated on an edge, then the sum (2.1) has at most two nonzero terms, namely those containing F_i associated to the adjacent vertices of this edge.

where

$$\begin{aligned} R_i(x,y) &= \frac{1}{2}(x_i-x)^2\frac{\partial^2\varphi}{\partial x^2}(c_i^1,c_i^2)+(x_i-x)(y_i-y)\frac{\partial^2\varphi}{\partial x\partial y}(c_i^1,c_i^2)\\ &+ \frac{1}{2}(y_i-y)^2\frac{\partial^2\varphi}{\partial y^2}(c_i^1,c_i^2),\\ S_i(x,y) &= (x_i-x)\frac{\partial^2\varphi}{\partial x^2}(d_i^1,d_i^2)+(y_i-y)\frac{\partial^2\varphi}{\partial x\partial y}(d_i^1,d_i^2),\\ T_i(x,y) &= (x_i-x)\frac{\partial^2\varphi}{\partial x\partial y}(e_i^1,e_i^2)+(y_i-y)\frac{\partial^2\varphi}{\partial y^2}(e_i^1,e_i^2), \end{aligned}$$

with $(c_i^1,c_i^2),(d_i^1,d_i^2),(e_i^1,e_i^2)$ situated 'between' the points (x,y) and $M_i(x_i,y_i)$. Replacing these expressions into (2.1) we obtain

$$\begin{aligned} &\mathcal{P}\varphi(x,y)\\ =\ &\varphi(x,y)\sum_{i=1}^V F_i(x,y)+\frac{\partial\varphi}{\partial x}(x,y)\left(\sum_{i=1}^V(x_i-x)F_i(x,y)+G_i(x,y)\right)\\ +\ &\frac{\partial\varphi}{\partial y}(x,y)\left(\sum_{i=1}^V(y_i-y)F_i(x,y)+H_i(x,y)\right)\\ +\ &\sum_{i=1}^V R_i(x,y)F_i(x,y)+S_i(x,y)G_i(x,y)+T_i(x,y)H_i(x,y). \end{aligned}$$

Using the properties (2.3),(2.4) and (2.5), we can write

$$\begin{aligned} &(\mathcal{P}\varphi)(x,y)-\varphi(x,y)\\ =\ &\sum_{i=1}^V R_i(x,y)F_i(x,y)+S_i(x,y)G_i(x,y)+T_i(x,y)H_i(x,y). \end{aligned}$$

Then, for all $i=1,\ldots,V$ we have

$$|R_i(x,y)|\le\frac{1}{2}K_2\left(|x-x_i|+|y-y_i|\right)^2.$$

If (x,y) is situated inside a triangle T, then the sum $\sum_{i=1}^V R_i(x,y)F_i(x,y)$ has at most three terms which are not zero, namely those corresponding to the vertices of the triangle T. In this case we can write

$$\sum_{i=1}^{V} |R_i(x,y)| \leq K_2 \sum_{i=1}^{3} (x - x_{\tau(i)})^2 + (y - y_{\tau(i)})^2 \leq 3K_2 L_{\max}^2,$$

where $(x_{\tau(i)}, y_{\tau(i)})$ are the vertices of the triangle T.
If (x,y) is situated on an edge E, then at least two terms of this sum are nonzero, namely those which correspond to the end-points of the edge E. In this case we have

$$\sum_{i=1}^{V} |R_i(x,y)| \leq 2K_2 L_{\max}^2.$$

Then, using Proposition 2.2 we obtain, in the case when (x,y) is situated inside a triangle T,

$$\begin{aligned} \sum_{i=1}^{V} |S_i(x,y)| \cdot |G_i(x,y)| &\leq \frac{1}{4} \sum_{i=1}^{3} |S_{\tau(i)}(x,y)| \cdot \max_{j=1,2,3} |x_{\tau(i)} - x_{\tau(j)}| \\ &\leq \frac{L_{\max}}{4} K_2 \sum_{i=1}^{3} |x - x_{\tau(i)}| + |y - y_{\tau(i)}| \\ &\leq \frac{3\sqrt{2}}{4} K_2 L_{\max}^2, \end{aligned}$$

and analogously

$$\sum_{i=1}^{V} |T_i(x,y)| \cdot |H_i(x,y)| \leq \frac{3\sqrt{2}}{4} K_2 L_{\max}^2.$$

When (x,y) is situated on an edge with the end-points M_k, M_j, we have $G_k(x,y) = H_j(x,y) = 0$.
Combining all the above inequalities, we finally obtain that

$$|(\mathcal{P}\varphi)(x,y) - \varphi(x,y)| \leq 3 \left(1 + \frac{\sqrt{2}}{2}\right) K_2 L_{\max}^2, \tag{2.8}$$

whence the conclusion.
Note that, if we refine the triangulation, then the rest of the interpolation formula (2.6) tends to zero. Actually, to improve the approximation, it is enough to refine only the two triangles of $\mathcal{T}$ which contain the greatest

edge.

3 A combined operator

We use now the idea of construction of Sheppard combined operators (see [1]) and construct the operator

$$(\mathcal{P}_1\varphi)(x,y)=\sum_{i=1}^{V}F_i(x,y)\left(\varphi(M_i)+\frac{\partial\varphi}{\partial x}(M_i)(x-x_i)+\frac{\partial\varphi}{\partial y}(M_i)(y-y_i)\right),$$

which is also preserves the functions 1, x and y. Using again the relations (2.8), (2.8) and (2.8), we obtain

$$\begin{aligned}
&-(\mathcal{R}_1\varphi)(x,y)\\
=\;&(\mathcal{P}_1\varphi)(x,y)-\varphi(x,y)\\
=\;&\sum_{i=1}^{V}F_i(x,y)\left(R_i(x,y)+(x-x_i)S_i(x,y)+(y-y_i)T_i(x,y)\right).
\end{aligned}$$

With the same arguments as in Theorem 2.3, we obtain

$$|(\mathcal{P}_1\varphi)(x,y)-\varphi(x,y)|\le 15K_2L_{\max}^2,$$

Note that this bound is greater than the one obtained in (2.8) for the operator $\mathcal{P}$.

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