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BIDIMENSIONAL INTERPOLATION OPERATORS OF FINITE ELEMENT TYPE AND DEGREE OF EXACTNESS TWO

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Abstract. For a given arbitrary triangulation of $\mathbb{R}^2$, we construct an interpolating operator which is exact for the polynomials in two variables of total degree ≤ 2. This operator is local, in the sense that the information around an interpolation node are taken from a small region around this point. We study the remainder of the interpolation formula.

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1. PRELIMINARIES

Given a set of V distinct points in $\mathbb{R}^2$, we construct a triangulation $\mathcal{T}$ and denote by $\Delta,\ \Delta \subseteq \mathbb{R}^2$, the set covered by the triangles of $\mathcal{T}$. Then, for each of the triangles in $\mathcal{T}$, some functions will be associated in the following way. Consider the triangle $T \in \mathcal{T}$ with the vertices $M_1(x_1, y_1),\ M_2(x_2, y_2),\ M_3(x_3, y_3)$ and the number

$$D_T = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

We define the functions $A, B, C : \mathbb{R}^2 \to \mathbb{R}$, depending on the triangle T,

$$\begin{aligned}
A(x,y) &= \tfrac{(x_3-x_2)(y-y_3)-(y_3-y_2)(x-x_3)}{D_T} \\
&= \tfrac{(x_3-x_2)(y-y_2)-(y_3-y_2)(x-x_2)}{D_T}, \\
B(x,y) &= \tfrac{(x_1-x_3)(y-y_1)-(y_1-y_3)(x-x_1)}{D_T} \\
&= \tfrac{(x_1-x_3)(y-y_3)-(y_1-y_3)(x-x_3)}{D_T}, \\
C(x,y) &= \tfrac{(x_2-x_1)(y-y_2)-(y_2-y_1)(x-x_2)}{D_T} \\
&= \tfrac{(x_2-x_1)(y-y_1)-(y_2-y_1)(x-x_1)}{D_T}
\end{aligned}$$

and $X, Y : \mathbb{R}^2 \to \mathbb{R}$,

$$X(x,y) = x, \quad Y(x,y) = y, \text{ for all } (x,y) \in \mathbb{R}^2.$$

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Definition 1. *The functions* $f^T_{M_1}, g^T_{M_1}, h^T_{M_1} : \mathbb{R}^2 \to \mathbb{R}$, *associated to the vertex* M_1 *of the triangle* T, *defined by*

$$\begin{aligned} f^T_{M_1} &= 2C^3 + 2B^3 - 3C^2 - 3B^2 + 1 + \alpha_1 ABC, \\ g^T_{M_1} &= (x_3 - x_1)(C^3 - CB^2 - 2C^2) + (x_2 - x_1)(B^3 - BC^2 - 2B^2) \\ &\quad + X - x_1 + \beta_1 ABC, \\ h^T_{M_1} &= (y_3 - y_1)(C^3 - CB^2 - 2C^2) + (y_2 - y_1)(B^3 - BC^2 - 2B^2) \\ &\quad + Y - y_1 + \gamma_1 ABC, \end{aligned}$$

are called shape functions. *Analogously, for the vertices* M_2 *and* M_3, *the shape functions are defined by circular permutations of the functions* A, B, C, *the parameters* $\alpha_1, \beta_1, \gamma_1$ *being replaced with* $\alpha_2, \beta_2, \gamma_2$ *for the vertex* M_2 *and respectively with* $\alpha_3, \beta_3, \gamma_3$ *for* M_3.

Some immediate properties of these functions are given in the following proposition.

Proposition 1. *The following statements are true for* $i, j \in \{1, 2, 3\}$.

1) $f^T_{M_i}(M_j) = \delta_{ij}$, $\dfrac{\partial f^T_{M_i}}{\partial x}(M_j) = 0$, $\dfrac{\partial f^T_{M_i}}{\partial y}(M_j) = 0$,
2) $g^T_{M_i}(M_j) = 0$, $\dfrac{\partial g^T_{M_i}}{\partial x}(M_j) = \delta ij$, $\dfrac{\partial g^T_{M_i}}{\partial y}(M_j) = 0$,
3) $h^T_{M_i}(M_j) = 0$, $\dfrac{\partial h^T_{M_i}}{\partial x}(M_j) = 0$, $\dfrac{\partial h^T_{M_i}}{\partial y}(M_j) = \delta_{ij}$,
4) *Along the edges of the triangle* $M_1M_2M_3$, *the functions* $f^T_{M_i}, g^T_{M_i}, h^T_{M_i}$ *depend only on the corresponding vertices.*

We also give some preliminary results, which will be used in the sequel. These results are summarized in the following proposition.

Proposition 2. *Let* $T = M_1M_2M_3$ *be a triangle and let* $f^T_{M_i}, g^T_{M_i}, h^T_{M_i}$ *be the shape functions defined in Definition* 1 *for* $i = 1, 2, 3$. *The following statements are true.*

1) $\sum\limits_{i=1}^{3} f^T_{M_i} = 1 - ABC(12 + \sum\limits_{i=1}^{3} \alpha_i)$,
2) $\sum\limits_{i=1}^{3} x_i f^T_{M_i} + g^T_{M_i} = X - ABC \sum\limits_{i=1}^{3} \beta_i + \alpha_i x_i + 4x_i$,
3) $\sum\limits_{i=1}^{3} y_i f^T_{M_i} + h^T_{M_i} = Y - ABC \sum\limits_{i=1}^{3} \gamma_i + \alpha_i y_i + 4y_i$,
4) $\sum\limits_{i=1}^{3} x_i^2 f^T_{M_i} + 2x_i g^T_{M_i} = X^2 - ABC\Big[6(x_1x_2 + x_2x_3 + x_3x_1) + \sum\limits_{i=1}^{3} 2\beta_i x_i + \alpha_i x_i^2 - 2x_i^2\Big]$,

5) $\sum\limits_{i=1}^{3} y_i^2 f_{M_i}^T + 2y_i h_{M_i}^T = Y^2 - ABC\Big[6(y_1y_2 + y_2y_3 + y_3y_1) + \sum\limits_{i=1}^{3} 2\gamma_i y_i + \alpha_i y_i^2 - 2y_i^2\Big],$

6) $\sum\limits_{i=1}^{3} x_iy_i f_{M_i}^T + y_i g_{M_i}^T + x_i h_{M_i}^T = XY - ABC\Big[3(x_2y_1 + x_1y_2 + x_3y_1 + x_1y_3 + x_2y_3 + x_3y_1) - 2\,(x_1y_1 + x_2y_2 + x_3y_3) + \sum\limits_{i=1}^{3} \gamma_i x_i + \beta_i y_i + \alpha_i x_i y_i\Big].$

Proof. The proof of the above formulas needs some elementary but quite long calculations which are not given here because of lack of space. □

2. THE INTERPOLATION OPERATOR

As in [3], to a given function $\varphi : \mathbb{R}^2 \to \mathbb{R}$, $\varphi \in C^1(\Delta)$, we associate the piecewise polynomial function $\mathcal{P}\varphi : \Delta \to \mathbb{R}$,

$$(1) \qquad (\mathcal{P}\varphi)(x,y) = \sum_{i=1}^{V} \varphi(M_i)F_i(x,y) + \tfrac{\partial\varphi}{\partial x}(M_i)G_i(x,y) + \tfrac{\partial\varphi}{\partial y}(M_i)H_i(x,y),$$

for $(x,y) \in \Delta$, where

$$F_i(x,y) = \begin{cases} f_{M_i}^T(x,y), & \text{if } (x,y) \text{ is inside or on the edges of the} \\ & \text{triangle } T \in \mathcal{T} \text{ which has } M_i \text{ as vertex,} \\ \quad 0, & \text{on the triangles which do not contain } M_i \end{cases}$$

and analogous definitions for the functions G_i and H_i.

The functions F_i, G_i, H_i are continuous on Δ due to the property 4) of Proposition 1, so the piecewise polynomial function $\mathcal{P}\varphi$ is continuous on Δ. In [3] we also gave the graphs of some particular functions $F_i, G_i.H_i$.

It is also immediate that $\mathcal{P}\varphi$ satisfies the interpolating conditions

$$(2) \qquad (\mathcal{P}\varphi)(M_i) = \varphi(M_i),\ \tfrac{\partial(\mathcal{P}\varphi)}{\partial x}(M_i) = \tfrac{\partial\varphi}{\partial x}(M_i),\ \tfrac{\partial(\mathcal{P}\varphi)}{\partial y}(M_i) = \tfrac{\partial\varphi}{\partial y}(M_i),$$

for $i = 1, \ldots, V$.

Other properties of the interpolating operator $\mathcal{P} : C^1(\Delta) \to C^1(\Delta)$, defined in (1), are given in the following proposition.

Proposition 3. *The operator $\mathcal{P}$ reproduces the polynomials in two variables of global degree two if and only if the following conditions are satisfied.*

$$\sum_{i=1}^{3} \alpha_i = -\,12,$$

$$\sum_{i=1}^{3} \alpha_i x_i + \beta_i = -\,4\sum_{i=1}^{3} x_i,$$

$$\sum_{i=1}^{3} \alpha_i y_i + \gamma_i = -\,4\sum_{i=1}^{3} y_i,$$

$$\sum_{i=1}^{3} \alpha_i x_i^2 + 2\beta_i x_i = -\,6(x_1x_2 + x_2x_3 + x_3x_1) + 2\sum_{i=1}^{3} x_i^2,$$
$$\sum_{i=1}^{3} \alpha_i y_i^2 + 2\gamma_i y_i = -\,6(y_1y_2 + y_2y_3 + y_3y_1) + 2\sum_{i=1}^{3} y_i^2,$$
$$\sum_{i=1}^{3} \alpha_i x_i y_i + \beta_i y_i + \gamma_i x_i = -\,3(x_2y_1 + x_1y_2 + x_3y_1 + x_1y_3 + x_2y_3 + x_3y_1) + 2\sum_{i=1}^{3} x_i y_i.$$

Proof. The proof is immediate if we use Proposition 2 and the fact that the expression $A(x,y)B(x,y)C(x,y)$ cannot be zero for all $(x,y) \in \Delta$. Also, we have to remark that, for a fixed $(x,y) \in \Delta$, the sum in (1) contains at most three nonzero terms, namely those corresponding to the triangle which contains the point (x,y). □

As an immediate consequence of this proposition we deduce the following result.

PROPOSITION 4. *Let $\mathcal{T}$ be a triangulation. If the parameters $\alpha_i, \beta_i, \gamma_i$, $i = 1,2,3$, are solutions of the system of equations given in Proposition 3, then the following identities are true for $(x,y) \in \Delta$:*

$$\sum_{i=1}^{V} F_i(x,y) = 1,$$
$$\sum_{i=1}^{V} (x_i - x)F_i(x,y) + G_i(x,y) = 0,$$
$$\sum_{i=1}^{V} (y_i - y)F_i(x,y) + H_i(x,y) = 0,$$
$$\sum_{i=1}^{V} (x_i - x)^2 F_i(x,y) + 2(x_i - x)G_i(x,y) = 0,$$
$$\sum_{i=1}^{V} (y_i - y)^2 F_i(x,y) + 2(y_i - y)H_i(x,y) = 0,$$
$$\sum_{i=1}^{V} (x_i - x)(y_i - y)F_i(x,y) + (y_i - y)G_i + (x_i - x)H_i(x,y) = 0.$$

The question now is if there exist such operators which reproduces the polynomials of global degree two. More precisely, we have to decide if the system of six equation, given in Proposition 3, with the unknowns $\alpha_i, \beta_i, \gamma_i,\ i = 1,2,3$, is solvable for all $x_i, y_i,\ i = 1,2,3$.

If we make the change of variables

$$\begin{aligned}
&\alpha_i = \lambda_i - 4,\ i = 1,2,3,\\
&\beta_1 = \mu_1 + 3(x_1 - x_2),\ \beta_2 = \mu_2 + 3(x_2 - x_3),\ \beta_3 = \mu_3 + 3(x_3 - x_1),\\
&\gamma_1 = \delta_1 + 3(y_1 - y_2),\ \gamma_2 = \delta_2 + 3(y_2 - y_3),\ \gamma_3 = \delta_3 + 3(y_3 - y_1),
\end{aligned}$$

the system becomes

$$\begin{aligned}
&\lambda_1 + \lambda_2 + \lambda_3 = 0,\\
&\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \mu_1 + \mu_2 + \mu_3 = 0,\\
&\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \delta_1 + \delta_2 + \delta_3 = 0,\\
&\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + 2(\mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3) = 0,\\
&\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 + 2(\delta_1 y_1 + \delta_2 y_2 + \delta_3 y_3) = 0,\\
&\lambda_1 x_1 y_1 + \lambda_2 x_2 y_2 + \lambda_3 x_3 y_3 + \mu_1 y_1 + \mu_2 y_2 + \mu_3 y_3 + \delta_1 x_1 + \delta_2 x_2 + \delta_3 x_3 = 0,
\end{aligned}$$

and is always solvable, being homogeneous. Moreover, this system is undetermined.

Remark 1. The same homogeneous system can be obtained making the change of variables

$$\begin{aligned}
&\alpha_i = \lambda_i - 4,\ i = 1,2,3,\\
&\beta_1 = \mu_1 + 3(x_1 - x_3),\ \beta_2 = \mu_2 + 3(x_2 - x_1),\ \beta_3 = \mu_3 + 3(x_3 - x_2),\\
&\gamma_1 = \delta_1 + 3(y_1 - y_3),\ \gamma_2 = \delta_2 + 3(y_2 - y_1),\ \gamma_3 = \delta_3 + 3(y_3 - y_2).
\end{aligned}$$

In the following we will restrict ourselves to the zero solution of the homogeneous system, more precisely to the operator $\mathcal{P}$ defined in (1), with

$$\begin{aligned}
&\alpha_1 = \alpha_2 = \alpha_3 = -4,\\
&\beta_1 = 3(x_1 - x_2),\ \beta_2 = 3(x_2 - x_3),\ \beta_3 = 3(x_3 - x_1),\\
&\gamma_1 = 3(y_1 - y_2),\ \gamma_2 = 3(y_2 - y_3),\ \gamma_3 = 3(y_3 - y_1).
\end{aligned}$$

In order to study the rest $\mathcal{R}\varphi$ of the interpolation formula

$$\varphi = \mathcal{P}\varphi + \mathcal{R}\varphi,$$

we need to establish the following result.

Lemma 1. *Let $T = M_1M_2M_3,\ M_i(x_i, y_i),\ i = 1,2,3,$ be a triangle. Then, for all $(x,y) \in T$, the following inequalities are true.*

1) $0 \le f_{M_1}^T(x,y) \le 1$,
2) $\left|g_{M_1}^T(x,y)\right| \le \max\left\{\frac{16}{27}|x_2 - x_1|, \frac{16}{27}|x_3 - x_1|\right\}$,
3) $\left|h_{M_1}^T(x,y)\right| \le \max\left\{\frac{16}{27}|y_2 - y_1|, \frac{16}{27}|y_3 - y_1|\right\}$.

Proof. 1) The inequalities were already proved in [2].

2) Using the equality $x - x_1 - (x_3 - x_1)C - (x_2 - x_1)B = 0$, the function $g_{M_1}^T$ can be written as

$$g_{M_1}^T = (1 - B - C)\left(rC(1 - C + B) + sB(1 - B - 2C)\right),$$

where $r = x_3 - x_1,\ s = x_2 - x_1$.

Case 1. $x_2 = x_3$. Denoting

$$\phi(B, C) = (1 - B - C)(C(1 - C) + B(1 - B) - BC),$$

we have to find the extremes of ϕ when $0 \le B \le 1,\ 0 \le C \le 1,\ B+C \le 1$. The stationary points of the function ϕ are $(B, C) \in \{(0, 1), (1, 0), (\rho, \rho)\}$, where ρ is a root of the equation $9\rho^2 - 7\rho + 1 = 0$, namely $\rho_{1,2} = (7 \pm \sqrt{13})/18$. The inequality $B + C \le 1$ is not satisfied by the stationary point (ρ_1, ρ_1). At the other stationary points, the function ϕ take the values $\phi(1, 0) = \phi(0, 1) = 0,\ \phi(\rho_2, \rho_2) = r \cdot (35 + 13\sqrt{13})/486$.

Then, as in [2],[1] we can prove that, on the edges of the triangle T, the function $g_{M_1}^T$ takes values between 0 and $\frac{4r}{27}$.

In conclusion, in this case we have $\left|g_{M_1}^T(x, y)\right| \le \frac{16}{27}|r|$, for all $(x, y) \in T$.

Case 2. $x_2 \ne x_3$. Making the transform $\omega : \mathbb{R}^2 \to \mathbb{R}^2$ described by the functions

$$u = u(x, y) = C - B,\ v = v(x, y) = B + C,\ (u, v) \in \mathbb{R}^2, \tag{3}$$

the triangle T maps into a domain denoted by U and the function $g_{M_1}^T(x, y) = g_{M_1}^T\left(\omega^{-1}(u, v)\right) = \psi(u, v)$ becomes

$$\psi(u, v) = \tfrac{1}{2}(1 - v)\left(r(v + u)(1 - u) + s(v - u)(1 - \tfrac{3v+u}{2})\right). \tag{4}$$

The stationary points of the function ψ are

$$(u, v) \in \{(1, 1), (-1, 1), (\rho(s - r), \rho(2r - s) + 1)\}, \tag{5}$$

where ρ is a root of the equation $3(4s^2 - 8rs + r^2)\rho^2 + 4(r - 2s)\rho + 1 = 0$, that is

$$\begin{aligned}
\rho_{1,2} &= \left(4s - 2r \mp \sqrt{4s^2 + 8rs + r^2}\right)^{-1},\ \text{if } 4s^2 - 8rs + r^2 \ne 0,\\
\rho &= \left(4(2s - r)\right)^{-1},\ \text{if } 4s^2 - 8rs + r^2 = 0.
\end{aligned}$$

We have to decide which of the stationary points (u, v) are situated inside the domain U. The stationary points $(1, 1)$ and $(-1, 1)$ are situated on the border of U and $\psi(1, 1) = \psi(-1, 1) = 0$.

[1]The restrictions to the edges are the same for our function and for the function considered in [2].

Subcase 2a). $4s^2-8rs+r^2\neq 0$. In this case, at the stationary points $P_{\rho_{1,2}}\left((s-r)\rho_{1,2},(2r-s)\rho_{1,2}+1\right)$, B and C have the values

$$\begin{aligned} B=\tfrac{v-u}{2} &= \tfrac{1}{2}\left(1+(3r-2s)\rho_{1,2}\right),\\ C=\tfrac{u+v}{2} &= \tfrac{1}{2}\left(1+r\rho_{1,2}\right). \end{aligned}$$

The condition that a stationary point P_ρ is situated inside the domain U is equivalent to $B\in[0,1],\ C\in[0,1],\ B+C\le 1$. In Appendix A we proved that, if P_ρ is situated inside U, then

$$|\psi(P_\rho)|\le\max\left\{\tfrac{16}{27}|r|,\tfrac{16}{27}|s|\right\},\ \text{for } \rho=\rho_{1,2}.$$

On the edges, we have

$$\begin{aligned} g^T_{M_1}|_{M_1M_2} &= s\,B(1-B)^2, \text{ and take values between 0 and } \tfrac{4s}{27},\\ g^T_{M_1}|_{M_1M_3} &= r\,C(1-C)^2, \text{ and take values between 0 and } \tfrac{4r}{27},\\ g^T_{M_1}|_{M_2M_3} &= 0. \end{aligned}$$

In conclusion, in this subcase we have

$$\left|g^T_{M_1}(x,y)\right|\le\max\left\{\frac{16|r|}{27},\frac{16|s|}{27}\right\}. \tag{6}$$

Subcase 2b). $4s^2-8rs+r^2=0$. In this case $\rho=(4(2s-r))^{-1}$, $s=\left(1\pm\frac{\sqrt{3}}{2}\right)r$ and at the stationary point $P_\rho\left((s-r)\rho,(2r-s)\rho+1\right)$, B and C take the values

$$B=\tfrac{2\pm\sqrt{3}}{8},\ C=\tfrac{7\pm\sqrt{3}}{16}.$$

The only stationary point situated inside U is $P\left(\frac{3+\sqrt{3}}{16},\frac{11-3\sqrt{3}}{16}\right)$, where the function ψ take the value $\psi(P)=\frac{19+11\sqrt{3}}{256}r\simeq r\cdot 0.1486\ldots$

On the edges, $|g^T_{M_1}|$ take values between 0 and $\left(1+\frac{\sqrt{3}}{2}\right)\frac{4}{27}|r|<\frac{8}{27}|r|$.

In conclusion, in this subcase we also have

$$\left|g^T_{M_1}(x,y)\right|\le\max\left\{\tfrac{16}{27}|r|,\tfrac{16}{27}|s|\right\}.$$

3) The proof is analogously with 2). □

3. THE INTERPOLATION FORMULA

In this section we study the interpolation formula

$$\varphi=\mathcal{P}\varphi+\mathcal{R}\varphi, \tag{7}$$

with $\mathcal{P}$ defined in (1), by proving the following theorem.

THEOREM 1. *Let $\varphi\in C^3(\operatorname{int}\Delta)$. Then we have*

$$\|\mathcal{R}\varphi\|_\infty=\sup_{(x,y)\in\Delta}|\varphi(x,y)-(\mathcal{P}\varphi)(x,y)|\le\left(\frac{32}{9}+\sqrt{2}\right)K_3L^3_{\max}, \tag{8}$$

where

$$K_3 = \sup_{(x,y)\in \mathrm{int}\,\Delta} \left\{ \left|\tfrac{\partial^3 \varphi}{\partial x^3}(x,y)\right|, \left|\tfrac{\partial^3 \varphi}{\partial x^2 \partial y}(x,y)\right|, \left|\tfrac{\partial^3 \varphi}{\partial x \partial y^2}(x,y)\right|, \left|\tfrac{\partial^3 \varphi}{\partial y^3}(x,y)\right| \right\} \tag{9}$$

and $L_{\max}$ is the length of the greatest edge of the triangles of $\mathcal{T}$.

Proof. We follow the ideas in [3] and write some Taylor formulas around the point $(x,y) \in \mathrm{int}\,\Delta$. For all $1 \le i \le V$ we have

$$\begin{aligned}
\varphi(M_i) &= \varphi(x,y) + (x_i - x)\tfrac{\partial \varphi}{\partial x}(x,y) + (y_i - y)\tfrac{\partial \varphi}{\partial y}(x,y) \\
&\quad + \tfrac{1}{2!}\left[(x_i - x)\tfrac{\partial}{\partial x} + (y_i - y)\tfrac{\partial}{\partial y}\right]^{(2)} \varphi(x,y) + R_i(x,y), \\
\tfrac{\partial \varphi}{\partial x}(M_i) &= \tfrac{\partial \varphi}{\partial x}(x,y) + (x_i - x)\tfrac{\partial^2 \varphi}{\partial x^2}(x,y) + (y_i - y)\tfrac{\partial^2 \varphi}{\partial x \partial y}(x,y) + S_i(x,y), \\
\tfrac{\partial \varphi}{\partial y}(M_i) &= \tfrac{\partial \varphi}{\partial y}(x,y) + (x_i - x)\tfrac{\partial^2 \varphi}{\partial x \partial y}(x,y) + (y_i - y)\tfrac{\partial^2 \varphi}{\partial y^2}(x,y) + T_i(x,y),
\end{aligned}$$

where

$$\begin{aligned}
R_i(x,y) &= \tfrac{1}{3!}\left[(x_i - x)\tfrac{\partial}{\partial x} + (y_i - y)\tfrac{\partial}{\partial y}\right]^{(3)} \varphi(c_i^1, c_i^2), \\
S_i(x,y) &= \tfrac{1}{2!}\left[(x_i - x)\tfrac{\partial}{\partial x} + (y_i - y)\tfrac{\partial}{\partial y}\right]^{(2)} \tfrac{\partial \varphi}{\partial x}(d_i^1, d_i^2), \\
T_i(x,y) &= \tfrac{1}{2!}\left[(x_i - x)\tfrac{\partial}{\partial x} + (y_i - y)\tfrac{\partial}{\partial y}\right]^{(2)} \tfrac{\partial \varphi}{\partial y}(e_i^1, e_i^2)
\end{aligned}$$

with $(c_i^1, c_i^2), (d_i^1, d_i^2), (e_i^1, e_i^2)$ situated 'between' the points (x,y) and $M_i(x_i, y_i)$. Replacing these expressions into (1) we obtain, using the identities given in Proposition 4,

$$\mathcal{P}\varphi(x,y) = \varphi(x,y) + \sum_{i=1}^{V} R_i(x,y)F_i(x,y) + S_i(x,y)G_i(x,y) + T_i(x,y)H_i(x,y).$$

On the other hand, for all $i = 1, \ldots, V$ we have

$$|R_i(x,y)| \le \tfrac{1}{6} K_3 \left(|x - x_i| + |y - y_i|\right)^3.$$

If (x,y) is situated inside a triangle T, then the sum $\sum_{i=1}^{V} R_i(x,y)F_i(x,y)$ has at most three terms which are not zero, namely those corresponding to the vertices of the triangle T. In this case we can write[2]

$$\sum_{i=1}^{V} |R_i(x,y)| \le \tfrac{\sqrt{2}}{3} K_3 \sum_{i=1}^{3} \left[(x - x_{\tau(i)})^2 + (y - y_{\tau(i)})^2\right]^{\frac{3}{2}} \le \sqrt{2} K_3 L_{\max}^3,$$

where $(x_{\tau(i)}, y_{\tau(i)})$ are the vertices of the triangle T.

If (x,y) is situated on an edge E, then at least two terms of this sum are nonzero, namely those which correspond to the end-points of the edge E. In

[2]We use the inequality $(a+b)^2 \le 2(a^2 + b^2)$ for $a = |x - x_{\tau(i)}|$, $b = |y - y_{\tau(i)}|$.

this case we have

$$\sum_{i=1}^{V} |R_i(x,y)| \leq \tfrac{2\sqrt{2}}{3} K_3 L_{\max}^3.$$

Further, for all $i = 1, \dots, V$ we have

$$|S_i(x,y)| \leq \tfrac{1}{2} K_3 \left(|x - x_i| + |y - y_i|\right)^2$$

and using Lemma 1 we obtain

$$\begin{aligned}
\sum_{i=1}^{V} |S_i(x,y)| \cdot |G_i(x,y)| &\leq \tfrac{16}{27} \sum_{i=1}^{3} |S_{\tau(i)}(x,y)| \cdot \max_{j=1,2,3} |x_{\tau(i)} - x_{\tau(j)}| \\
&\leq \tfrac{16}{27} L_{\max} K_3 \sum_{i=1}^{3} (x - x_{\tau(i)})^2 + (y - y_{\tau(i)})^2 \\
&\leq \tfrac{16}{9} K_3 L_{\max}^3,
\end{aligned}$$

and analogously

$$\sum_{i=1}^{V} |T_i(x,y)| \cdot |H_i(x,y)| \leq \tfrac{16}{9} K_3 L_{\max}^3,$$

for (x,y) situated inside a triangle T. When (x,y) is situated on an edge with the end-points M_k, M_j, we have

$$G_k(x,y) = H_j(x,y) = 0.$$

Combining all the above inequalities, we finally obtain that

$$|(\mathcal{P}\varphi)(x,y) - \varphi(x,y)| \leq \left(\tfrac{32}{9} + \sqrt{2}\right) K_3 L_{\max}^3, \tag{10}$$

whence the conclusion. □

4. APPENDIX

With the notations of Lemma 1, we have to decide which of the stationary points P_{ρ_1} and P_{ρ_2}, $\rho_{1,2} = \left(4s - 2r \mp \sqrt{4s^2 + 8rs + r^2}\right)^{-1}$ are situated inside the domain U.

We consider two cases.

Case 1. $s \neq 0$. Denoting $t = \frac{r}{s}$, condition $B \in [0,1]$ reduces to

$$-1 \leq \tfrac{3t-2}{4-2t-\sqrt{t^2+8t+4}} \leq 1,$$

for ρ_1 when $s > 0$ and for ρ_2 when $s < 0$ (case A)

$$-1 \leq \tfrac{3t-2}{4-2t+\sqrt{t^2+8t+4}} \leq 1,$$

for ρ_1 when $s < 0$ and for ρ_2 when $s > 0$ (case B) and is fulfilled when

$$t \in \left[-4+2\sqrt{3}, 0\right] \cup \left[\tfrac{17-\sqrt{97}}{12}, \infty\right], \text{ in case A}$$
$$t \in \left[-4+2\sqrt{3}, \tfrac{17+\sqrt{97}}{12}\right], \text{ in case B.}$$

Condition $C \in [0,1]$ reduces to

$$-1 \le \tfrac{t}{4-2t-\sqrt{t^2+8t+4}} \le 1, \text{in case A}$$
$$-1 \le \tfrac{t}{4-2t+\sqrt{t^2+8t+4}} \le 1, \text{in case B}$$

and is fulfilled when

$$t \in \left(-\infty, -4-2\sqrt{3}\right] \cup \left[-4+2\sqrt{3}, \tfrac{4-\sqrt{10}}{2}\right] \cup \left[\tfrac{3}{4}, \infty\right) \text{ in case A,}$$
$$t \in \left(-\infty, -4-2\sqrt{3}\right] \cup \left[-4+2\sqrt{3}, \tfrac{4+\sqrt{10}}{2}\right] \text{ in case B.}$$

Finally, condition $B+C \le 1$ means

$$\tfrac{2t-1}{4-2t-\sqrt{t^2+8t+4}} \le 0, \text{ in case A,}$$
$$\tfrac{2t-1}{4-2t+\sqrt{t^2+8t+4}} \le 0, \text{ in case B.}$$

and is satisfied for

$$t \in \left(-\infty, -4-2\sqrt{3}\right] \cup \left[-4+2\sqrt{3}, \tfrac{1}{2}\right] \cup \left[4-2\sqrt{3}, \infty\right) \text{ in case A,}$$
$$t \in \left(-\infty, -4-2\sqrt{3}\right] \cup \left[-4+2\sqrt{3}, \tfrac{1}{2}\right] \cup \left[4+2\sqrt{3}, \infty\right) \text{ in case B.}$$

Intersecting all the intervals and denoting $I_1 = \left[-4+2\sqrt{3}, 0\right], I_2 = \left[\frac{3}{4}, \infty\right), I_3 = \left[-4+2\sqrt{3}, \frac{1}{2}\right]$, we find:

$$\begin{aligned} &\text{If } s>0, \text{ then} && P_{\rho_1} \in U \text{ for } t \in I_1 \cup I_2, \\ & && P_{\rho_2} \in U \text{ for } t \in I_3. \\ &\text{If } s<0, \text{ then} && P_{\rho_1} \in U \text{ for } t \in I_3, \\ & && P_{\rho_2} \in U \text{ for } t \in I_1 \cup I_2. \end{aligned}$$

Then, the values of ψ at the stationary points are

$$\psi\left(\rho(s-r), \rho(2r-s)+1\right) = -\tfrac{1}{4}(s-2r)^2\rho\left(\rho^2(r^2-8rs+4s^2)+\rho(2r-4s)+1\right),$$

$\rho = \rho_{1,2}$. Using the equality $\rho^2(4s^2-8rs+r^2) = -\frac{1}{3}\left(1+4\rho(r-2s)\right)$, we find

$$\psi(P_\rho) = -\tfrac{1}{6}(s-2r)^2\rho\left[1+\rho(r-2s)\right],$$

meaning

$$\psi(P_{\rho_{1,2}}) = \tfrac{1}{6}(s-2r)\tfrac{(1-2t)(t-2\pm\sqrt{t^2+8t+4})}{(2t-4\pm\sqrt{t^2+8t+4})^2}.$$

We also need the inequalities

$$|s-2r| \le 3 \cdot \max\{|r|, |s|\}, \text{ if } \tfrac{r}{s} < 0, \tag{11}$$
$$|s-2r| \le 2 \cdot \max\{|r|, |s|\}, \text{ if } \tfrac{r}{s} > 0. \tag{12}$$

Defining $\xi_1 : I_1 \cup I_2 \to \mathbb{R}$,

$$\xi_1(t) = \frac{(1-2t)(t-2+\sqrt{t^2+8t+4})}{(2t-4+\sqrt{t^2+8t+4})^2},$$

we have

$$\xi_1'(t) = \frac{3(8+t-3\sqrt{t^2+8t+4})}{(4-2t-\sqrt{t^2+8t+4})^3} > 0$$

so ξ_1 will be increasing on I_1 and on I_2. Since

$$\xi_1(-4+2\sqrt{3}) = \frac{\sqrt{3}-5}{16},\ \xi_1(0) = 0,\ \xi_1\left(\frac{3}{4}\right) = -\frac{16}{9},\ \lim_{t\to\infty} \xi_1(t) = -\frac{4}{9},$$

using (11) and (12) we finally obtain

$$|\psi(P_{\rho_1})| \leq \max\left\{\frac{16}{27}|r|, \frac{16}{27}|s|\right\}. \tag{13}$$

Further, defining $\xi_2 : I_3 \to \mathbb{R}$,

$$\xi_2(t) = \frac{(1-2t)(t-2-\sqrt{t^2+8t+4})}{(2t-4-\sqrt{t^2+8t+4})^2},$$

we have

$$\xi_2'(x) = \frac{3(8+t+3\sqrt{t^2+8t+4})}{(4-2t+\sqrt{t^2+8t+4})^3} > 0$$

and therefore the function is increasing on I_3. Since

$$\xi_2(-4+2\sqrt{3}) = \frac{\sqrt{3}-5}{16},\ \xi_2(0) = -\frac{1}{9},\ \xi_2(\tfrac{1}{2}) = 0,$$

using again (11) and (12) we have

$$|\psi(P_{\rho_2})| \leq \frac{5-\sqrt{3}}{32}\max\{|r|,|s|\} < \max\left\{\frac{16}{27}|r|, \frac{16}{27}|s|\right\}. \tag{14}$$

Case 2. $s = 0$. In this case ρ is a root of the equation $3r^2\rho^2 + 4r\rho + 1 = 0$, meaning $\rho_1 = -1/r,\ \rho_2 = -1/(3r)$. Conditions $B \in [0,1],\ C \in [0,1]$ and $B + C \in [0,1]$ are satisfied only by the stationary point P_{ρ_2} and the value of ψ at this point is $\psi(P_{\rho_2}) = 4r/27$, whence $|\psi(P_{\rho_2})| < \frac{16}{27}|r|$.

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