REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 32, N° 1, 2003, pp. 109–120

SOME SUFFICIENT CONDITIONS FOR THE CONVERGENCE OF THE CASCADE ALGORITHM AND FOR THE CONTINUITY OF THE SCALING FUNCTION*

DANIELA ROŞCA†

Abstract. We define a class of matrices which includes, under some natural assumptions, the matrices $\mathbf{m}(0)$, $\mathbf{m}(1)$ and T_{2N-1}, which are the key matrices of the wavelet theory. The matrices of this class have the property that the eigenvalues of a product matrix are products of their eigenvalues. This property is used in establishing some sufficient conditions for the convergence of the cascade algorithm and some sufficient conditions for the continuity of the scaling function. We generalize here the particular results obtained by us in a previous paper.

MSC 2000. 39A13, 65F30, 42C40.

Keywords. Dilation equation, scaling function, cascade algorithm, wavelets.

1. INTRODUCTION

The dilation equation plays an important role in wavelets theory. A dilation equation is a functional equation having the form

$$\phi(t) = 2\sum_{k=0}^{N} h_k \phi(2t-k), \quad \text{with } h_k \in \mathbb{R},\ k = 0,\dots,N. \tag{1}$$

Any nonzero solution ϕ of such an equation is called a *scaling function*. The scaling functions lead to wavelets: if ϕ is a scaling function, then the associated "mother wavelet" is defined as

$$\psi(t) = 2\sum_{k=0}^{N} (-1)^k h_{N-k}\phi(2t-k).$$

The dilation equation is linear, so any multiple of a solution is a solution. It is convenient to normalize so that

$$\int_{-\infty}^{\infty} \phi(t)\,\mathrm{d}t = 1.$$

*This work was supported by CNCSIS under grant 33532/2003, code 418/A4.

†Technical University of Cluj-Napoca, Dept. of Mathematics, str. Daicoviciu 15, 400020 Cluj-Napoca, Romania, e-mail: Daniela.Catinas@math.utcluj.ro.

This relation implies

$$h_0 + h_1 + \ldots + h_N = 1,$$

so in this paper we will suppose this condition satisfied. Also (see [3]), if this condition is satisfied, then the dilation equation (1) has a unique and compactly supported solution $\phi(t)$. This solution may be a distribution.

For the coefficients h_k some other conditions are required. In the wavelets literature these are the so-called $\mathbf{A_p}$ conditions.

Definition 1. *Let $p \in \mathbb{N}^*$. We say that the coefficients h_k of the dilation equation* (1) *satisfy the **Condition** $\mathbf{A_p}$ if*

$$\sum_{k=0}^{N} (-1)^k k^m h_k = 0, \quad \text{for } m = 0, \ldots, p-1,$$

with the convention $0^0 = 1$.

A way to solve the dilation equation is the cascade algorithm described by

$$\phi^{i+1}(t) = 2\sum_{k=0}^{N} h_k \phi^i(2t-k), \ i = 0, 1, \ldots, \tag{2}$$

with $\phi^0(t)$ usually taken as the box function

$$\phi^0(t) = \begin{cases} 1, & t \in [0,1], \\ 0, & \text{otherwise}. \end{cases}$$

The scaling function ϕ is the limit

$$\phi = \lim_{i\to\infty} \phi^i.$$

In Section 2 we study some classes of matrices that will help us in studying the convergence of the cascade algorithm (Section 3) and then in studying the continuity of the scaling function (Section 4).

2. A CLASS OF MATRICES

In this section we present the classes $\mathcal{C}_N$ of matrices of $\mathcal{M}_N(\mathbb{R})$, $N \in \mathbb{N}^*$, which are closed with respect to the operation of multiplication and have the property that the eigenvalues of a product matrix are products of their eigenvalues. These classes include the matrices T_{2N-1}, $\mathbf{m}(0)$ and $\mathbf{m}(1)$ given in (15) , (3) and (4), which are the key matrices in studying the dilation equation.

For a given $N \in \mathbb{N}$ we consider the matrix $S = (s_{ij})_{1\le i,j\le N}$ with

$$s_{ij} = \begin{cases} \binom{j-1}{i-1}, & i \le j, \\ 0, & i > j. \end{cases}$$

A simple calculation shows that its inverse is $S^{-1} = (u_{ij})_{1\le i,j\le N}$ with

$$u_{ij} = \begin{cases} (-1)^{i+j}\binom{j-1}{i-1}, & i \leq j, \\ 0, & i > j. \end{cases}$$

Then we define the class $\mathcal{C}_N$ as the set of the matrices $M \in \mathcal{M}_N(\mathbb{R})$ with the property that the matrix SMS^{-1} is lower triangular. The set $\mathcal{C}_N$ has the following properties:

LEMMA 1. *Let* M_1, $M_2 \in \mathcal{C}_N$. *Then:*

1. $M_1 M_2 \in \mathcal{C}_N$;
2. *The eigenvalues of the product matrix* $M_1 M_2$ *are products of the eigenvalues of* M_1 *and* M_2.

Proof. It is immediately that the matrices M and SMS^{-1} have the same eigenvalues. Denote λ_k^i, $k = 1, \ldots, N$, the eigenvalues of M_i, such that

$$SM_iS^{-1} = \begin{bmatrix} \lambda_1^i & & & 0 \\ \times & \lambda_2^i & & \\ \vdots & \vdots & \ddots & \\ \times & \times & \cdots & \lambda_N^i \end{bmatrix}, \quad \text{for } i \in \{1, 2\}.$$

Then,

$$S(M_1M_2)S^{-1} = \left(SM_1S^{-1}\right)\left(SM_2S^{-1}\right) = \begin{bmatrix} \lambda_1^1\lambda_1^2 & & & 0 \\ \times & \lambda_2^1\lambda_2^2 & & \\ \vdots & \vdots & \ddots & \\ \times & \times & \cdots & \lambda_N^1\lambda_N^2 \end{bmatrix},$$

whence both conclusions 1 and 2 follow. □

3. THE CONTINUITY OF THE SCALING FUNCTION

As shown in [3, ch. 7], the study of the continuity of the scaling function involves the matrices $\mathbf{m}(0)$ and $\mathbf{m}(1)$ defined by

$$(\mathbf{m}(0))_{ij} = 2h_{2i-j-1}, \quad 1 \leq i, j \leq N, \tag{3}$$

$$(\mathbf{m}(1))_{ij} = 2h_{2i-j}, \quad 1 \leq i, j \leq N. \tag{4}$$

In the following we will prove that $\mathbf{m}(0) \in \mathcal{C}_N$ and $\mathbf{m}(1) \in \mathcal{C}_N$. First we have to establish some preliminary results.

LEMMA 2. *Let* $i \in \mathbb{N}^*$. *If the coefficients* h_k *satisfy the condition* $\mathbf{A_i}$, *then*

$$\sum_{k=i-1}^{N} \binom{k}{i-1}(-1)^k h_k = 0 \quad \text{and} \quad \sum_{k=i-2}^{N} \binom{k+1}{i-1}(-1)^k h_k = 0. \tag{5}$$

Proof. Both $\binom{k}{i-1}$ and $\binom{k+1}{i-1}$ are polynomials in k of degree $i-1$. Thus, relations (5) hold if the condition $\mathbf{A_i}$ is satisfied. □

LEMMA 3. *Let $p \in \mathbb{N}^*$. Then the following relations hold:*

$$(6)\qquad \sum_{m=0}^{p} \binom{2p}{2m}\binom{k+m-1}{2p-1} = \binom{2k-1}{2p-1}, \quad k = 2p,\ 2p+1,\ldots$$

$$(7)\qquad \sum_{m=1}^{p} \binom{2p}{2m-1}\binom{k+m-2}{2p-1} = \binom{2k-2}{2p-1}, \quad k = 2p, \quad 2p+1,\ldots$$

$$(8)\qquad \sum_{m=0}^{p} \binom{2p+1}{2m}\binom{k+m-1}{2p} = \binom{2k-1}{2p}, \quad k = 2p+1,\ 2p+2,\ldots$$

$$(9)\qquad \sum_{m=1}^{p+1} \binom{2p+1}{2m-1}\binom{k+m-2}{2p} = \binom{2k-2}{2p}, \quad k = 2p+1,\ 2p+2,\ldots$$

Proof. To prove (6) we show that the polynomials

$$P(x) = \sum_{m=0}^{p} \binom{2p}{2m} (x+m-1)(x+m-2)\ldots(x+m-(2p-1))$$

and

$$Q(x) = (2x-1)(2x-2)\ldots(2x-(2p-1))$$

(of degree $2p-1$) coincide. Then they will coincide at the points $x = 2p,\ 2p+1,\ldots$.

In order to prove that $P = Q$, we see first that the coefficient of x^{2p-1} in both P and Q is 2^{2p-1}. Then we prove that they have the same roots. It is immediately that $P(j) = Q(j) = 0$, for $j = 1, 2, \ldots, p-1$. So, it remains to prove that $P((2s-1)/2) = 0$, for $s = 1, 2, \ldots, p$. Denoting

$$(10)\quad R(x,t) = \sum_{m=0}^{p} \binom{2p}{2m} (x+m-1)(x+m-2)\ldots(x+m-t), \text{ for } t \in \mathbb{N}^*,$$

we may write $R(x, 2p-1) = P(x)$.

Thus, we have to prove that $R((2s-1)/2, 2p-1) = 0$, for $s = 1, 2, \ldots, p$. For this, we write a recurrence formula as follows. First we easily deduce that

$$R(x,t) = R(x-1,t) + tR(x-1,t-1).$$

Then, by induction on n we immediately obtain

(11)

$$\begin{aligned} R(x,t) =& R(x-n,t) + \binom{n}{1} tR(x-n,t-1) + \binom{n}{2} t(t-1)R(x-n,t-2) + \ldots \\ &+ \binom{n}{i} t(t-1)\ldots(t-i+1)R(x-n,t-i) + \ldots \\ &+ \binom{n}{n} t(t-1)\ldots(t-n+1)R(x-n,t-n), \end{aligned}$$

for $1 \le n < \min(x,t)$.

So $R(x,t)$ is combination of $R(x-n,t)$, $R(x-n,t-1),\ldots,R(x-n,t-n)$. Writing the relation (11) for $t=2p-1$, $x=(2s-1)/2$, $s=2,3,\ldots,p$ and $n=s-1$, we get

$$R\left(\tfrac{2s-1}{2},2p-1\right)=R\left(\tfrac{1}{2},2p-1\right)+\binom{s-1}{1}(2p-1)\,R\left(\tfrac{1}{2},2p-2\right)+\ldots \\ +\binom{s-1}{s-1}(2p-1)\ldots(2p-s+1)\,R\left(\tfrac{1}{2},2p-s\right), \tag{12}$$

for $s=2,\ldots,p$.

In order to have $R\left(\frac{2s-1}{2},2p-1\right)=0$, for $s=1,\ldots,p$, it is enough to prove that

$$R\left(\tfrac{1}{2},p+t\right)=0, \quad \text{for } t=0,1,\ldots,p-1.$$

Evaluating $R\left(\frac{1}{2},p\right)$, we obtain

$$\begin{aligned} R\left(\tfrac{1}{2},p\right) &= \sum_{m=0}^{p}\binom{2p}{2m}\left(\tfrac{1}{2}+m-1\right)\ldots\left(\tfrac{1}{2}+m-p\right) \\ &= \tfrac{1}{2^p}\sum_{m=0}^{p}\binom{2p}{2m}(2m-1)(2m-3)\ldots(2m-2p+1). \end{aligned}$$

Consider now the polynomial

$$H(x)=\sum_{m=0}^{p}\binom{2p}{2m}(2m-1)(2m-3)\ldots(2m-2p+1)\,x^m.$$

A simple induction on m shows that

$$\frac{\binom{2p}{2m}(2m-1)\ldots(2m-2p+1)}{(-1)^p(2p-1)!!}=(-1)^m\binom{p}{m}, \quad \text{for } m=0,1,\ldots,p.$$

Thus, the polynomial H gets the expression

$$H(x)=(-1)^p(2p-1)!!\sum_{m=0}^{p}(-1)^m\binom{p}{m}x^m=(-1)^p(2p-1)!!\,(1-x)^p,$$

whence $H(1)=H'(1)=\ldots=H^{(p-1)}(1)=0$.

In conclusion we have $0=H(1)=2^pR\left(\frac{1}{2},p\right)$, and further, for $t=1,\ldots,p-1$,

$$\begin{aligned} &H^{(t)}(1)= \\ &=\sum_{m=t}^{p}\binom{2p}{2m}(2m-1)(2m-3)\ldots(2m-2p+1)\,m(m-1)\ldots(m-t+1) \\ &=\sum_{m=0}^{p}\binom{2p}{2m}(2m-1)(2m-3)\ldots(2m-2p+1)\,m(m-1)\ldots(m-t+1) \\ &=0. \end{aligned}$$

Now, evaluating $R\left(\frac{1}{2}, p+t\right)$, from (10) follows that

$$\begin{aligned} R(1/2, p+t) &= \tfrac{1}{2^{p+t}} \sum_{m=0}^{p} \tbinom{2p}{2m} (2m-1)(2m-3)\dots(2m-2p+1) \cdot \\ &\quad \cdot \left[(2m-2p-1)\dots(2m-2p-2t+1) \right] \\ &= \tfrac{1}{2^{p+t}} \sum_{m=0}^{p} \tbinom{2p}{2m} (2m-1)(2m-3)\dots(2m-2p+1) \left[a_0 + \right. \\ &\quad \left. + a_1 m + a_2 m(m-1) + \dots a_t m(m-1)\dots(m-t+1) \right] \\ &= \tfrac{1}{2^{p+t}} \left(a_0 H(1) + a_1 H'(1) + \dots + a_t H^{(t)}(1) \right) = 0, \end{aligned}$$

for $t = 1, \dots, p-1$. Using now (12), we obtain

$$R\left(\tfrac{2s-1}{2}, 2p-1\right) = 0, \quad \text{for } s = 1, \dots, p.$$

In conclusion the polynomials P and Q coincide, fact which proves the relation (6). Analogously we can prove the relations (7), (8) and (9). □

Now we prove the following theorem.

Theorem 1. *If the coefficients h_k satisfy the condition $\mathbf{A_{N-1}}$, then $\mathbf{m}(0) \in \mathcal{C}_N$ and $\mathbf{m}(1) \in \mathcal{C}_N$.*

Proof. We have to show that $S\mathbf{m}(0)S^{-1} = (\alpha_{ij})_{1 \le i,j \le N}$ and $S\mathbf{m}(1)S^{-1} = (\beta_{ij})_{1 \le i,j \le N}$ are lower triangular. A simple calculation shows that

$$\alpha_{ij} = 2 \sum_{l=1}^{j} (-1)^{l+j} \tbinom{j-1}{l-1} \sum_{k=i} \tbinom{k-1}{i-1} h_{2k-l-1}, \quad 1 \le i, j \le N, \tag{13}$$

$$\beta_{ij} = 2 \sum_{l=1}^{j} (-1)^{l+j} \tbinom{j-1}{l-1} \sum_{k=i} \tbinom{k-1}{i-1} h_{2k-l}, \quad 1 \le i, j \le N. \tag{14}$$

In order to have lower triangular matrices we have to prove that $\alpha_{ij} = \beta_{ij} = 0$, for $i = 1, \dots, N-1$, $j = i+1, \dots, N$, equalities which reduce to

$$\sum_{l=1}^{j} (-1)^{l} \tbinom{j-1}{l-1} \sum_{k=i} \tbinom{k-1}{i-1} h_{2k-l-1} = 0, \quad i = 1, \dots, N-1, \; j = i+1, \dots, N,$$

$$\sum_{l=1}^{j} (-1)^{l} \tbinom{j-1}{l-1} \sum_{k=i} \tbinom{k-1}{i-1} h_{2k-l} = 0, \quad i = 1, \dots, N-1, \; j = i+1, \dots, N.$$

First we prove these relations for fixed i, $1 \le i < N$ and $j = i+1$ (Step 1). Then we use the induction to prove them for arbitrary j, $j \le N$ (Step 2).

Step 1. Let $1 \le i < N$. We prove that $\alpha_{i,i+1} = \beta_{i,i+1} = 0$, which means

$$\begin{aligned}\sum_{l=1}^{i+1}(-1)^l\binom{i}{l-1}\sum_{k=i}\binom{k-1}{i-1}h_{2k-l-1} &= 0,\\ \sum_{l=1}^{i+1}(-1)^l\binom{i}{l-1}\sum_{k=i}\binom{k-1}{i-1}h_{2k-l} &= 0.\end{aligned}$$

Consider two cases:

Case 1. Even i: $i = 2p$. In this case

$$\begin{aligned}&-\frac{\alpha_{i,i+1}}{2} =\\ &= \sum_{l=1}^{2p+1}(-1)^l\binom{2p}{l-1}\sum_{k=2p}\binom{k-1}{2p-1}h_{2k-l-1}\\ &= \sum_{l=1,\, l\,\mathrm{odd}}^{2p+1} + \sum_{l=1,\, l\ \mathrm{even}}^{2p+1}\\ &= \sum_{m=1}^{p}\binom{2p}{2m-1}\sum_{k=2p}\binom{k-1}{2p-1}h_{2k-2m-1} - \sum_{m=0}^{p}\binom{2p}{2m}\sum_{k=2p}\binom{k-1}{2p-1}h_{2k-2m-2}\\ &= \sum_{m=1}^{p}\binom{2p}{2m-1}\sum_{k'=2p-m}\binom{k'+m-1}{2p-1}h_{2k'-1} - \sum_{m=0}^{p}\binom{2p}{2m}\sum_{k'=2p-m}\binom{k'+m-1}{2p-1}h_{2k'-2}.\end{aligned}$$

With the convention $\binom{2p}{q} = 0$ whenever $q > 2p$, we can replace $\sum_{k'=2p-m}$ with $\sum_{k'=p}$. Then, using formulas (6) and (7) we further obtain

$$\begin{aligned}\frac{\alpha_{i,i+1}}{2} &= \sum_{k=p}\left(\sum_{m=0}^{p}\binom{2p}{2m}\binom{k+m-1}{2p-1}\right)h_{2k-2} - \sum_{k=p}\left(\sum_{m=1}^{p}\binom{2p}{2m-1}\binom{k+m-1}{2p-1}\right)h_{2k-1}\\ &= \sum_{k=p}\binom{2k-1}{2p-1}h_{2k-2} - \sum_{k=p}\binom{2k}{2p-1}h_{2k-1}\\ &= \sum_{k=i-2}^{N}(-1)^k\binom{k+1}{i-1}h_k\\ &= 0.\end{aligned}$$

Analogously we can prove that

$$\beta_{i,i+1} = -2\sum_{k=i-1}^{N}(-1)^k\binom{k}{i-1}h_k = 0.$$

Case 2. Odd i: $i = 2p + 1$. In this case, the same arguments can be used in order to prove that $\alpha_{i,i+1} = \beta_{i,i+1} = 0$. Also, at this step we prove that

$\beta_{1,j} = 0$ for $j > 2$. Indeed, evaluating $\beta_{1,j}$ we obtain

$$\tfrac{1}{2}\beta_{1,j} = \sum_{l=1}^{j} (-1)^{l+j} \binom{j-1}{l-1} \sum_{k=1} h_{2k-l} = \tfrac{1}{2} (-1)^j \sum_{l=1}^{j} (-1)^l \binom{j-1}{l-1} = 0.$$

Step 2. We use the induction method, so let us suppose that

$$\begin{cases} \alpha_{ij} = 0, & \text{for } 1 \le i < N-1,\ i < j < N, \\ \beta_{ij} = \beta_{i-1,j} = 0, & \text{for } 2 \le i < N-1,\ i < j < N \end{cases}$$

and prove that $\alpha_{i,j+1} = 0$ and $\beta_{i,j+1} = 0$. A simple calculation show that $\beta_{i,j+1} = \beta_{ij} + \alpha_{ij} = 0$ and $\alpha_{i,j+1} = \beta_{ij} + \beta_{i-1,j} = 0$. Thus, the induction method allows us to state that $\alpha_{ij} = \beta_{ij} = 0$ for $1 \le i < N$, $i < j < N$. This means that the matrices $S\mathbf{m}(0)S^{-1}$ and $S\mathbf{m}(1)S^{-1}$ are lower triangular. Let us mention that the condition $\mathbf{A_i}$ is enough to have zeros on the row i, above the diagonal. □

Now, let us turn back to the continuity.

The columns of $\mathbf{m}(0)$ and $\mathbf{m}(1)$ add to 1 if we suppose the condition $\mathbf{A_1}$ satisfied. If $\mathbf{e} = [1 \dots 1]$, then $\mathbf{e}\,\mathbf{m}(0) = \mathbf{e}$ and $\mathbf{e}\,\mathbf{m}(1) = \mathbf{e}$. The dilation equation in the vector form, on the interval $[0,1)$, is:

$$\mathbf{\Phi}(t) = \mathbf{m}(0)\,\mathbf{\Phi}(2t) + \mathbf{m}(1)\,\mathbf{\Phi}(2t-1), \text{ where}$$

$\mathbf{\Phi}(t) = [\phi(t)\ \ \phi(t+1) \dots \phi(t+N-1)]^{\mathrm{t}}$. The first digit t_1 in $t = 0.t_1t_2t_3\dots$ (written in the base 2) decides whether the recursion use $\mathbf{m}(0)$ or $\mathbf{m}(1)$:

$$\mathbf{\Phi}(t) = \mathbf{m}(t_1)\,\mathbf{\Phi}(0.t_2t_3\dots).$$

Further,

$$\mathbf{\Phi}(t) = \mathbf{m}(t_1)\,\mathbf{m}(t_2)\,\mathbf{\Phi}(0.t_3t_4\dots).$$

A nearby point T begins with the same digits. At some step, the digits differ. If $T = 0.t_1t_2T_3T_4\dots$, then

$$\mathbf{\Phi}(t) - \mathbf{\Phi}(T) = \mathbf{m}(t_1)\,\mathbf{m}(t_2)\left[\mathbf{\Phi}(0.t_3t_4\dots) - \mathbf{\Phi}(0.T_3T_4\dots)\right].$$

To prove the continuity on $[0,1)$ means to show that $\mathbf{\Phi}(t)$ is close to $\mathbf{\Phi}(T)$ when t and T share more digits $t_1, t_2, \dots, t_k$. Also, this should happen outside the interval $[0,1)$ (for more details, see [3]). Actually one works with the matrices $\mathbf{m}_{N-1}(0)$ and $\mathbf{m}_{N-1}(1)$ of order $N-1$. These matrices are restrictions of the linear operators $\mathbf{m}(0)$ resp. $\mathbf{m}(1)$ to the vector spaces perpendicular to the vector $\mathbf{e} = [1 \dots 1]$. They may be determined in the following way:

Let us define

$$U = \begin{bmatrix} \mathbf{I}_{N-1} & \mathbf{0}_{N-1} \\ -\mathbf{e}_{N-1} & 1 \end{bmatrix}.$$

If $\mathbf{m}(0)$ (resp. $\mathbf{m}(1)$) is the block matrix

$$B = \begin{bmatrix} \mathbf{a}_{N-1} & \mathbf{b}_{N-1} \\ \mathbf{c}_{N-1} & d \end{bmatrix},$$

then, multiplying $U^{-1}BU$ by blocks, we get

$$\begin{bmatrix} \mathbf{I}_{N-1} & \mathbf{0}_{N-1} \\ \mathbf{e}_{N-1} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{a}_{N-1} & \mathbf{b}_{N-1} \\ \mathbf{c}_{N-1} & d \end{bmatrix} \begin{bmatrix} \mathbf{I}_{N-1} & \mathbf{0}_{N-1} \\ -\mathbf{e}_{N-1} & 1 \end{bmatrix} = \begin{bmatrix} B_{N-1} & \mathbf{b}_{N-1} \\ \mathbf{0}_{N-1} & 1 \end{bmatrix},$$

with $B_{N-1} = \mathbf{a}_{N-1} - \mathbf{b}_{N-1}\mathbf{e}_{N-1}$. The matrix B_{N-1} will be the above mentioned restriction $\mathbf{m}_{N-1}(0)$ of $\mathbf{m}(0)$ (resp. $\mathbf{m}_{N-1}(1)$ of $\mathbf{m}(1)$). It is easy to see that the eigenvalues of $\mathbf{m}_{N-1}(t_1)$, $t_1 \in \{0, 1\}$, are the same as the eigenvalues of $\mathbf{m}(t_1)$, after removing the eigenvalue $\lambda = 1$.

The following proposition show the relation between the eigenvalues of the products $\mathbf{m}(t_1)\,\mathbf{m}(t_2)$, $t_i \in \{0, 1\}$, and the eigenvalues of the products $\mathbf{m}_{N-1}(t_1)\,\mathbf{m}_{N-1}(t_2)$.

Proposition 1. *The eigenvalues of the product* $\mathbf{m}(t_1)\,\mathbf{m}(t_2)$, $t_i \in \{0, 1\}$ *are* $\lambda = 1$ *and the eigenvalues of the product* $\mathbf{m}_{N-1}(t_1)\,\mathbf{m}_{N-1}(t_2)$.

Proof. The matrices $U^{-1}\mathbf{m}(t_i)\,U$ and $\mathbf{m}(t_i)$ have the same eigenvalues. Denoting

$$\mathbf{m}(t_i) = \begin{bmatrix} \mathbf{a}^i_{N-1} & \mathbf{b}^i_{N-1} \\ \mathbf{c}^i_{N-1} & d^i \end{bmatrix},$$

we have

$$\mathbf{m}(t_i) = U \begin{bmatrix} \mathbf{m}_{N-1}(t_i) & \mathbf{b}^i_{N-1} \\ \mathbf{0}_{N-1} & 1 \end{bmatrix} U^{-1},$$

with $\mathbf{m}_{N-1}(t_i) = \mathbf{a}^i_{N-1} - \mathbf{b}^i_{N-1}\mathbf{e}_{N-1}$. Then,

$$\begin{aligned} \mathbf{m}(t_1)\,\mathbf{m}(t_2) &= U \begin{bmatrix} \mathbf{m}_{N-1}(t_1) & \mathbf{b}^1_{N-1} \\ \mathbf{0}_{N-1} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{m}_{N-1}(t_2) & \mathbf{b}^2_{N-1} \\ \mathbf{0}_{N-1} & 1 \end{bmatrix} U^{-1} \\ &= U \begin{bmatrix} \mathbf{m}_{N-1}(t_1)\,\mathbf{m}_{N-1}(t_2) & \mathbf{m}_{N-1}(t_1)\,\mathbf{b}^2_{N-1} + \mathbf{b}^1_{N-1} \\ \mathbf{0}_{N-1} & 1 \end{bmatrix} U^{-1}, \end{aligned}$$

whence the conclusion. □

The relation between the eigenvalues of the matrices $\mathbf{m}_{N-1}(0)$ and $\mathbf{m}_{N-1}(1)$ and the continuity of the scaling function is stated in the following theorem (see [1, p. 1042]).

Theorem 2 (Daubechies and Lagarias, 1991). *Assume that the coefficients* h_k, $k = 0, \ldots, N$, *satisfy*

$$\sum_{k=0}^{N} (-1)^k\, h_k = 0.$$

If $\rho(\mathbf{m}_{N-1}(0), \mathbf{m}_{N-1}(1)) < 1$, *then there exists a continuous nontrivial solution of the dilation equation* (1)*.*

Here $\rho(A_1, A_2)$ denotes the "generalized spectral radius" of the matrices A_1 and A_2,

$$\rho(A_1, A_2) = \limsup_{n\to\infty} \left[\max_{d_j=1 \text{ or } 2} \rho(A_{d_1} \dots A_{d_n})^{1/n} \right],$$

with $\rho(A) = \max\left\{ |\mu| : \ \mu \text{ eigenvalue of } A \right\}$ the spectral radius of A.

Combining the result of this theorem with the result stated in the previous proposition, we can prove the following theorem.

THEOREM 3. *Consider the dilation equations* (1) *and condition* $\mathbf{A_{N-1}}$ *satisfied. If* $h_0 \in \left(-\frac{1}{2} + \frac{1}{2^{N-1}}, \frac{1}{2}\right)$, *then the scaling function* ϕ *is continuous.*

Proof. Since the matrices $\mathbf{m}(0)$ and $\mathbf{m}(1)$ belong to the class $\mathcal{C}_N$, we can precise the eigenvalues of an arbitrary product $\mathbf{m}(t_1)\mathbf{m}(t_2)\dots\mathbf{m}(t_n)$, for $t_i \in \{0,1\}$, if the eigenvalues of the matrices $\mathbf{m}(0)$ and $\mathbf{m}(1)$ are known. If we impose the condition that all the eigenvalues of these two matrices have the modulus less than 1, then $\rho(\mathbf{m}_{N-1}(0), \mathbf{m}_{N-1}(1)) < 1$ (using Proposition 1), whence the continuity of the scaling function. The eigenvalues of $\mathbf{m}(0)$ are $1, \frac{1}{2}, \frac{1}{2^2}, \dots, \frac{1}{2^{N-2}}, 2h_0$ and the eigenvalues of $\mathbf{m}(1)$ are $1, \frac{1}{2}, \frac{1}{2^2}, \dots, \frac{1}{2^{N-2}}, 2h_N$ (see [3]). Thus, we have to impose the conditions $|2h_0| < 1$ and $|2h_N| < 1$. Condition $\mathbf{A_{N-1}}$ implies $h_0 + h_N = \frac{1}{2^{N-1}}$, so finally we must have $h_0 \in \left(-\frac{1}{2} + \frac{1}{2^{N-1}}, \frac{1}{2}\right)$. □

4. THE CONVERGENCE OF THE CASCADE ALGORITHM

In this section we give some conditions on the coefficients h_k for the convergence of the cascade algorithm, namely for the convergence in L^2 of the sequence $\left\{\phi^i\right\}_i$ defined in (2) to a L^2-function ϕ.

In the study of the L^2-convergence the following matrix is involved:

$$T = (\downarrow 2)\, 2HH^{\mathrm{t}}, \tag{15}$$

where H is the Toepliz matrix with h_k on the k-th diagonal: $H_{ij} = h_{i-j}$.

As shown in [3], the cascade algorithm for $\phi(t)$ becomes the power method $a^{(i+1)} = Ta^{(i)}$ for the equation $Ta = a$ (with $a^{(i)}$, $a^{(i+1)}$, a vectors, the components of $a^{(i)}$ being $a^{(i)}(k) = \int_{-\infty}^{\infty} \phi^{(i)}(t)\, \phi^{(i)}(t+k)\, \mathrm{d}t$). In [3] the following theorem is proved (Theorem 7.7, p. 239).

THEOREM 4. *The infinite matrix* $T = (\downarrow 2)\, 2HH^t$ *and its submatrix* T_{2N-1} *always has* $\lambda = 1$ *as eigenvalue. The power iteration* $a^{(i+1)} = T_{2N-1}a^{(i)}$ *converges to the eigenvector* $T_{2N-1}a = a$ *if and only if* T_{2N-1} *satisfies*
Condition E: *The matrix* T_{2N-1} *has all the eigenvalues satisfying* $|\lambda| < 1$, *except for a simple eigenvalue at* $\lambda = 1$.

So, **Condition E** is the key to iteration of filters and thus to wavelets. If it is satisfied, then the cascade algorithm converges to a Riesz basis $\{\phi(t-k)\}_{k\in\mathbb{Z}}$.

Using this theorem we can establish the following result.

THEOREM 5. *Consider the dilation equation* (1) *satisfying the condition* $\mathbf{A_{N-1}}$. *The cascade algorithm converges in* L^2 *if and only if*

$$h_0 \in \left(\tfrac{1}{2^N}\left(1-\sqrt{2^{2N-2}-1}\right), \tfrac{1}{2^N}\left(1+\sqrt{2^{2N-2}-1}\right)\right).$$

Proof. We will show that $T_{2N-1} \in \mathcal{C}_{2N-1,}$ then we determine its eigenvalues and use Theorem 4.

The entries of the matrix T_{2N-1} are $(T_{2N-1})_{ij} = 2t_{2i-j}$, where t_i are the coefficients of the polynomial $G(z) = z^N H(z) H\left(z^{-1}\right) = \sum_{k=0}^{2N} t_k z^k$, where $H(z) = \sum_{k=0}^{N} h_k z^k$. Condition $\mathbf{A_{N-1}}$ implies that

$$H(z) = (z+1)^{N-1}(h_N z + h_0),$$

since it is equivalent to

$$H(-1) = H'(-1) = \ldots = H^{(N-2)}(-1) = 0.$$

Then, $G(z) = (z+1)^{2N-2}(h_0 z + h_N)(h_N z + h_0)$. It is immediately that

$$G(-1) = G'(-1) = \ldots = G^{(2N-3)}(-1) = 0,$$

relations which are equivalent to

$$\sum_{k=0}^{2N}(-1)^k t_k = \sum_{k=1}^{2N}(-1)^k k t_k = \ldots = \sum_{k=1}^{2N}(-1)^k k^{2N-3} t_k = 0.$$

Thus, the coefficients t_k satisfy the condition $\mathbf{A_{2N-2}}$. On the other hand we have

$$\sum_{k=0}^{2N} t_k = G(1) = H(1)^2 = \left(\sum_{k=0}^{N} h_k\right)^2 = 1.$$

The matrix T_{2N-1} will belong to the class $\mathcal{C}_{2N-1}$, due to the same arguments that were used in Theorem 3.3 to prove that $\mathbf{m}(1) \in \mathcal{C}_N$. So, the eigenvalues of the matrix T_{2N-1}will be $\lambda_i = 2^{-i+1}$, $i = 1, \ldots, 2N-2$ and $\lambda_{2N-1} = \left(ST_{2N-1}S^{-1}\right)_{2N-1,2N-1}$. As in (14), we find that

$$\left(ST_{2N-1}S^{-1}\right)_{ij} = 2\sum_{l=1}^{j}(-1)^{l+j}\binom{j-1}{l-1}\sum_{k=i}^{2N-1}\binom{k-1}{i-1}t_{2k-l},$$

whence, for $i = j = 2N-1$, we obtain

$$\begin{aligned}
\lambda_{2N-1} &= 2\sum_{l=1}^{2N-1}(-1)^{l+1}\binom{2N-2}{l-1}\sum_{k=2N-1}^{2N-1}\binom{k-1}{2N-2}t_{2k-l}\\
&= 2\sum_{l=2N-2}^{2N-1}(-1)^{l+1}\binom{2N-2}{l-1}t_{2(2N-1)-l}\\
&= 2\left(-\binom{2N-2}{2N-3}t_{2N}+\binom{2N-2}{2N-2}t_{2N-1}\right)\\
&= -2t_0\left(2N-2\right)+2t_1.
\end{aligned}$$

Evaluating t_0 and t_1 we obtain

$$\begin{aligned}
t_0 &= h_0h_N,\\
t_1 &= h_0h_{N-1}+h_1h_N\\
&= h_0\left(h_0+h_N\left(N-1\right)\right)+\left(h_0\left(n-1\right)+h_N\right)h_N\\
&= h_0^2+2h_0h_N\left(N-1\right)+h_N^2.
\end{aligned}$$

So,

$$\begin{aligned}
\tfrac{\lambda_{2N-1}}{2} &= -h_0h_N\left(2N-2\right)+h_0^2+2h_0h_N\left(N-1\right)+h_N^2\\
&= h_0^2+h_N^2\\
&= h_0^2+\left(2^{-N+1}-h_0\right)^2.
\end{aligned}$$

Condition E reduces to $\lambda_{2N-1}<1$, which is equivalent to

$$h_0\in\left(\tfrac{1}{2^N}\left(1-\sqrt{2^{2N-2}-1}\right),\tfrac{1}{2^N}\left(1+\sqrt{2^{2N-2}-1}\right)\right). \qquad \square$$

REFERENCES

[1] Daubechies, I. and Lagarias, J., *Two-scale Difference Equations II. Local Regularity, Infinite Products of Matrices and Fractals*, SIAM J. Math. Anal., **23**, no. 4, pp. 1031–1079, 1992.

[2] Roşca, D., *Some classes of matrices with applications in wavelet theory*, in Mathematical Analysis and Approximation Theory, Burg Verlag, pp. 233–242, 2002.

[3] Strang, G. and Nguyen, T., *Wavelets and Filter Banks*, Wellesley-Cambridge Press, 1996.

Received by the editors: November 27, 2002.