On a possible determination of the frame bounds

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1 Introduction.

In a separable Hilbert space $\mathcal{H}$, a subset $\{e_n, n \in \mathbb{N}\}$ is called a *frame* if there exist A, B, $B > 0$, $A < \infty$ (called the frame bounds) such that $B \|x\|^2 \leqslant \sum_{n\in\mathbb{N}} |\langle x, e_n\rangle|^2 \leqslant A \|x\|^2$, for every $x \in \mathcal{H}$. For such a sequence, we can find the set $\{\tilde{e}_n, n \in \mathbb{N}\}$ (called the *dual frame*) having the bounds A^{-1}, B^{-1}, and allowing the reconstruction $x = \sum_{n\in\mathbb{N}} \langle x, e_n\rangle \tilde{e}_n = \sum_{n\in\mathbb{N}} \langle x, \tilde{e}_n\rangle e_n$, for every $x \in \mathcal{H}$ (see [2]). The advantage of the frames over the orthonormal and complete bases (which allow the Fourier expansion $x = \sum_{n\in\mathbb{N}} \langle x, e_n\rangle e_n$, $\forall x \in \mathcal{H}$) is that the set $\{e_n, n \in \mathbb{N}\}$ need be neither orthonormal nor linearly independent. Moreover, if $A = B$ (tight frame), than that frame allows the unique expansion $x = A^{-1} \sum_{n\in\mathbb{N}} \langle x, e_n\rangle e_n$, $\forall x \in \mathcal{H}$, similarly to the Fourier one.

We give conditions which ensure that the subset $\{x_n, n \in \mathbb{N}^*\}$ of a separable real Hilbert space $\mathcal{H}$ is a frame, and we obtain formulas for the frame bounds in terms of the eigenvalues of the Gram matrices of the finite subsets.

2 Preliminaries.

In this section we remind some known relations which we shall use in the following.

Let $\{x_n, n \in \mathbb{N}^*\}$ be a subset of the separable real Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, $x \in \mathcal{H}$ and $\langle\cdot,\cdot\rangle_e$ the standard Euclidean product in $\mathbb{R}^n$. The Gram matrices associated to $\{x_n, n \in \mathbb{N}^*\}$, defined by

$$G_n = G(x_1, ..., x_n) = \begin{pmatrix} \langle x_1, x_1\rangle & \langle x_1, x_2\rangle & ... & \langle x_1, x_n\rangle \\ \langle x_2, x_1\rangle & \langle x_2, x_2\rangle & ... & \langle x_2, x_n\rangle \\ \vdots & & & \\ \langle x_n, x_1\rangle & \langle x_n, x_2\rangle & ... & \langle x_n, x_n\rangle \end{pmatrix}, \quad n \in \mathbb{N}^*,$$

have the following properties:

P1: All the eigenvalues of the matrices G_n are nonnegative numbers; if the set $\{x_1, ..., x_n\}$ is linearly independent, then these eigenvalues are positive.

P2: The system

$$\left\{\begin{array}{l} c_1^n \langle x_1, x_1\rangle + c_2^n \langle x_1, x_2\rangle + ... + c_n^n \langle x_1, x_n\rangle = \langle x_1, x\rangle \\ \vdots \\ c_1^n \langle x_n, x_1\rangle + c_2^n \langle x_n, x_2\rangle + ... + c_n^n \langle x_n, x_n\rangle = \langle x_n, x\rangle \end{array}\right. \tag{2.1}$$

with the unknowns $c_1^n, c_2^n, ..., c_n^n$, is solvable for every $x \in \mathcal{H}$, since if a row of the matrix of coefficients is a linear combination of the other rows, then the same thing happens in the augmented matrix.

If $\operatorname{rank} G_n = p = p(n)$, (for example $x_{\tau_n(1)}, ..., x_{\tau_n(p)}$ are linearly independent, where τ_n is a permutation of the set $\{1, 2, ..., n\}$, and $x_{\tau_n(p+1)}, ..., x_{\tau_n(n)}$ are linear combinations of them), then we will consider the solution $(c_1^n, c_2^n, ..., c_n^n)$, with $\left(c_{\tau_n(1)}^n, ..., c_{\tau_n(p)}^n\right)$ as the solution of the linear system

$$G_{\tau_n(p)} \begin{pmatrix} c_1^n \\ \vdots \\ c_p^n \end{pmatrix} = \begin{pmatrix} \left\langle x_{\tau_n(1)}, x\right\rangle \\ \vdots \\ \left\langle x_{\tau_n(p)}, x\right\rangle \end{pmatrix}, \tag{2.2}$$

and $c_{\tau_n(p+1)}^n = ... = c_{\tau_n(n)}^n = 0$. (We have denoted $G_{\tau(p)} = G\left(x_{\tau(1)}, ..., x_{\tau(p)}\right)$).

Denoting by $\lambda_n^{\min}$ and $\lambda_n^{\max}$ the smallest, respective the largest eigenvalue of the Gram matrix G_n, then the following inequalities hold:

P3: $\lambda_n^{\min} \langle y, y\rangle_e \leq \langle G_n y, y\rangle_e \leq \lambda_n^{\max} \langle y, y\rangle_e, \ \forall\ y \in \mathbb{R}^n$.

P4: $\lambda_n^{\min} \langle G_n y, y\rangle_e \leq \langle G_n y, G_n y\rangle_e \leq \lambda_n^{\max} \langle G_n y, y\rangle_e, \ \forall\ y \in \mathbb{R}^n$.

Proof. $\langle G_n y, G_n y\rangle_e - \lambda_n^{\max} \langle G_n y, y\rangle_e = \langle G_n^2 y, y\rangle_e - \langle \lambda_n^{\max} G_n y, y\rangle_e = \langle (G_n^2 - \lambda_n^{\max} G_n)\, y, y\rangle_e$.

It can be easily proved that if A is a symmetric matrix having the diagonal form B, $\operatorname{diag} B = \left(\lambda_n^{\min}, \ldots, \lambda_n^{\max}\right)$, and P is a polynomial, then the matrix $P(A)$ has the diagonal form C, with $\operatorname{diag} C = \left(P\left(\lambda_n^{\min}\right), \ldots, P\left(\lambda_n^{\max}\right)\right)$.

Hence the diagonal form of the matrix $G_n^2 - \lambda_n^{\max} G_n$ is

$$\operatorname{diag}\left(G_n^2 - \lambda_n^{\max} G_n\right) = \left(\lambda_n^{\min}\left(\lambda_n^{\min} - \lambda_n^{\max}\right), ..., \lambda_n^{\max}\left(\lambda_n^{\max} - \lambda_n^{\max}\right)\right).$$

All its eigenvalues are nonpositive numbers, so $\langle (G_n^2 - \lambda_n^{\max} G_n)\, y, y\rangle_e$ is a negative definite quadratic form, whence the stated inequality.

In the same way, $\langle G_n y, G_n y\rangle_e - \lambda_n^{\min} \langle G_n y, y\rangle_e = \left\langle \left(G_n^2 - \lambda_n^{\min} G_n\right) y, y\right\rangle_e$. The diagonal form of the matrix $G_n^2 - \lambda_n^{\min} G_n$ is

$$\operatorname{diag}\left(G_n^2 - \lambda_n^{\min} G_n\right) = \left(\lambda_n^{\min}\left(\lambda_n^{\min} - \lambda_n^{\min}\right), ..., \lambda_n^{\max}\left(\lambda_n^{\max} - \lambda_n^{\min}\right)\right).$$

All its eigenvalues are nonnegative numbers, so $\left\langle \left(G_n^2 - \lambda_n^{\min} G_n\right) y, y\right\rangle_e$ is a positive definite quadratic form, whence the stated inequality. $\square$

We will study the set $\{x_n : n \in \mathbb{N}^*\}$, which may be linearly dependent.

3 Properties of the sum $\langle x_1, x\rangle^2 + ... + \langle x_n, x\rangle^2$, $x \in \mathcal{H}$, $x \neq 0$.

Consider in the beginning a fixed $n \in \mathbb{N}^*$.

Let $C_n = (c_1^n, c_2^n, ..., c_n^n)$ and $X_n = \begin{pmatrix} \langle x_1, x\rangle \\ \vdots \\ \langle x_n, x\rangle \end{pmatrix}$. The system (2.1) may be written as $G_n C_n^T = X_n$.

If we suppose that $x_{\tau_n(1)}, ..., x_{\tau_n(p)}$ are linearly independent and $x_{\tau_n(k)}$, $k = \overline{p+1, n}$, are linear combinations of them ($p = p(n)$), the solution of the system (2.1), considered in **P2**, may be written as:

$$\left\{ \begin{array}{l} C_{p(n)}^T = G_{p(n)}^{-1} X_{p(n)} \\ c_{\tau_n(p+1)}^n = ... = c_{\tau_n(n)}^n = 0, \end{array} \right. \tag{3.1}$$

where

$$G_{p(n)} = G\left(x_{\tau_n(1)}, ..., x_{\tau_n(p)}\right) \text{ and } X_{p(n)} = \begin{pmatrix} \left\langle x_{\tau_n(1)}, x\right\rangle \\ \vdots \\ \left\langle x_{\tau_n(p)}, x\right\rangle \end{pmatrix}. \tag{3.2}$$

Consider the expression

$$E(X_n) = c_1^n \langle x_1, x\rangle + ... + c_n^n \langle x_n, x\rangle = C_n X_n.$$

In the case when $(c_1^n, \ldots, c_n^n)$ is the solution (3.1) of the system (2.1), it becomes the quadratic form

$$E\left(X_{p(n)}\right) = C_n X_n = \left\langle G_{p(n)}^{-1} X_{p(n)}, X_{p(n)}\right\rangle_e.$$

The eigenvalues of the matrix $G_{p(n)}^{-1}$ are the inverses of the eigenvalues of $G_{p(n)}$, (which are positive numbers), so, taking into account **P3**, we shall have that:

$$E\left(X_{p(n)}\right) = \left\langle G_{p(n)}^{-1} X_{p(n)}, X_{p(n)}\right\rangle_e \leq \frac{1}{\lambda_n^{\min}(p)} \left\langle X_{p(n)}, X_{p(n)}\right\rangle_e, \tag{3.3}$$

$\lambda_n^{\min}(p)$ being the smallest eigenvalue of the matrix $G_{p(n)}$.

Remark 1: Instead of the linearly independent elements $x_{\tau_n(1)}, ..., x_{\tau_n(p)}$, we may take other linearly independent elements $x_{\sigma_n(1)}, ..., x_{\sigma_n(p)}$ with σ_n permutation of $\{1, 2, ..., n\}$, different from τ_n. Hence, the matrix $G_{p(n)}$ is not unique, so it is possible to find more values for $\lambda_n^{\min}(p)$. We will choose the largest of them.

Consequences:

1. Let $(c_1^n, \ldots, c_n^n)$ be an arbitrary solution of the system (2.1). According to **P4** we obtain:

$$\begin{aligned}
\sum_{k=1}^{n} \langle x_k, x\rangle^2 &= \sum_{k=1}^{n} \left(c_1^n \langle x_1, x\rangle + ... + c_n^n \langle x_n, x\rangle\right)^2 \\
&= \left\langle G_n C_n^T, G_n C_n^T \right\rangle_e \leq \lambda_n^{\max} \left\langle G_n C_n^T, C_n^T \right\rangle_e \\
&= \lambda_n^{\max} \sum_{k=1}^{n} c_k^n \left(c_1^n \langle x_k, x_1\rangle + c_2^n \langle x_k, x_2\rangle + ... + c_n^n \langle x_k, x_n\rangle\right) \\
&= \lambda_n^{\max} \sum_{k=1}^{n} c_k^n \langle x_k, x\rangle ,
\end{aligned} \tag{3.4}$$

Remark 2: When $p(n) = n$, the above inequality can be immediately obtained: from **P3** it follows that

$$\frac{1}{\lambda_n^{\max}} \langle X_n, X_n\rangle_e \leq \left\langle G_n^{-1} X_n, X_n \right\rangle_e = E(X_n), \quad \text{i.e. } \sum_{k=1}^{n} \langle x_k, x\rangle^2 \leq \lambda_n^{\max} \sum_{k=1}^{n} c_k^n \langle x_k, x\rangle .$$

2. From (3.3) we obtain:

$$\langle X_n, X_n\rangle \geq \left\langle X_{p(n)}, X_{p(n)} \right\rangle \geq \lambda_n^{\min}(p)\, E\left(X_{p(n)}\right), i.e.$$

$$\sum_{k=1}^{n} \langle x_k, x\rangle^2 \geq \sum_{k=1}^{p} \left\langle x_{\tau_n(k)}, x\right\rangle^2 \geq \lambda_n^{\min}(p) \sum_{k=1}^{p} c_{\tau_n(k)}^n \left\langle x_{\tau_n(k)}, x \right\rangle \tag{3.5}$$

where $(c_1^n, ..., c_n^n)$ is the solution of the system (2.1) defined by (3.1).

4 Properties of the sum $\sum_{k=1}^{n} c_k \langle x_k, x\rangle , x \in \mathcal{H}, x \neq 0, (c_1, ..., c_n)$ solution of the system (2.1).

Suppose a fixed $n \in \mathbb{N}^*$ and p defined at the beginning of section 2.

Theorem 1. Let $F_n : \mathbb{R}^n \to \mathbb{R}$, $F_n(t_1, \ldots, t_n) = \left\langle x - \sum_{k=1}^{n} t_k x_k, x - \sum_{k=1}^{n} t_k x_k \right\rangle$, and $(c_1^n, \ldots, c_n^n)$ an arbitrary solution of the system (2.1). Then:

1°. If $p = n$, then $\min F_n = \langle x, x\rangle - \sum_{k=1}^{n} c_k^n \langle x_k, x\rangle$.

2°. If $p < n$, then $\min F_n = \langle x, x\rangle - \sum_{k=1}^{p} c_{\tau_n(k)}^n \left\langle x_{\tau_n(k)}, x\right\rangle = \langle x, x\rangle - \sum_{k=1}^{n} c_k^n \langle x_k, x\rangle$.

Proof.

1°. The necessary conditions for extremum, $\frac{\partial F_n}{\partial t_k} = 0$, $k = \overline{1,n}$, lead to the system (2.1). The Hessian matrix of the function F_n is, at any point, the Gram matrix $G(x_1, ..., x_n)$, which is strictly positive defined cf. **P1**. So, F_n will have a minimum attained at $(c_1^n, ..., c_n^n)$, the solution of the system (2.1), namely:

$$\min F_n = \langle x, x \rangle - \sum_{k=1}^{n} c_k^n \langle x_k, x \rangle .$$

2°. First we prove the following auxiliary results.

Lemma 1. Let $\left(c_{\tau_n(1)}^n, \ldots, c_{\tau_n(p)}^n\right)$ be the solution of the system (2.2) and $(d_1^n, \ldots, d_n^n)$ an arbitrary solution of the system (2.1).

Then $\sum\limits_{k=1}^{n} d_k^n \langle x_k, x \rangle = \sum\limits_{k=1}^{p} c_{\tau_n(k)}^n \left\langle x_{\tau_n(k)}, x \right\rangle$, i.e. the value of the function F_n is the same at every stationary point: $\langle x, x \rangle - \sum\limits_{k=1}^{p} c_{\tau_n(k)}^n \left\langle x_{\tau_n(k)}, x \right\rangle$.

Proof. For the sake of simplicity we omit the upper indices and we consider that $x_1, \ldots, x_p$ are linearly independent and $x_{p+1}, \ldots, x_n$ are linear combinations of them:

$$x_{p+j} = \sum_{k=1}^{p} \alpha_{jk} x_k, \quad j = \overline{1, n-p}, \quad \text{with } \alpha_{jk} \in \mathbb{R}.$$

Then the solution of the system (2.2) can be written as $(c_1, \ldots, c_p)$.

The system (2.1) becomes:

$$\begin{cases} \langle x_1, x_1 \rangle d_1 + \ldots + \langle x_1, x_p \rangle d_p = \langle x, x_1 \rangle - \sum\limits_{j=1}^{n-p} d_{p+j} \sum\limits_{k=1}^{p} \alpha_{jk} \langle x_1, x_k \rangle \\ \vdots \\ \langle x_p, x_1 \rangle d_1 + \ldots + \langle x_p, x_p \rangle d_p = \langle x, x_p \rangle - \sum\limits_{j=1}^{n-p} d_{p+j} \sum\limits_{k=1}^{p} \alpha_{jk} \langle x_p, x_k \rangle \\ \vdots \\ \langle x_n, x_1 \rangle d_1 + \ldots + \langle x_n, x_p \rangle d_p = \langle x, x_n \rangle - \sum\limits_{j=1}^{n-p} d_{p+j} \sum\limits_{k=1}^{p} \alpha_{jk} \langle x_n, x_k \rangle \end{cases}$$

Its general solution will be $(d_1, \ldots, d_n)$, with $d_i = \frac{\det A_i}{\det G_p}$, $i = \overline{1,p}$ and $d_{p+1}, \ldots, d_n$ arbitrary, the matrices A_i, $i = \overline{1,p}$, being given by

$$A_i = \begin{pmatrix} \langle x_1, x_1 \rangle \ldots \langle x_1, x_{i-1} \rangle & \langle x, x_1 \rangle - \sum\limits_{j=1}^{n-p} d_{p+j} \sum\limits_{k=1}^{p} \alpha_{jk} \langle x_1, x_k \rangle & \langle x_1, x_{i+1} \rangle \ldots \langle x_1, x_p \rangle \\ \vdots & & \\ \langle x_p, x_1 \rangle \ldots \langle x_p, x_{i-1} \rangle & \langle x, x_p \rangle - \sum\limits_{j=1}^{n-p} d_{p+j} \sum\limits_{k=1}^{p} \alpha_{jk} \langle x_p, x_k \rangle & \langle x_p, x_{i+1} \rangle \ldots \langle x_p, x_p \rangle \end{pmatrix}$$

We get

$$\sum_{k=1}^{n} d_k \langle x_k, x\rangle = \sum_{i=1}^{p} d_i \langle x_i, x\rangle + \sum_{i=1}^{n-p} d_{p+i} \langle x_{p+i}, x\rangle =$$
$$= \frac{1}{\det G_p} \sum_{i=1}^{p} (\det A_i) \langle x_i, x\rangle + \sum_{j=1}^{n-p} d_{p+j} \sum_{k=1}^{p} \alpha_{jk} \langle x_k, x\rangle .$$

If we split $\det A_i$ after the column i we obtain:

$$\det A_i = \begin{vmatrix} \langle x_1, x_1\rangle \ldots \langle x_1, x_{i-1}\rangle & \langle x, x_1\rangle & \langle x_1, x_{i+1}\rangle \ldots \langle x_1, x_p\rangle \\ & \vdots & \\ \langle x_p, x_1\rangle \ldots \langle x_p, x_{i-1}\rangle & \langle x, x_p\rangle & \langle x_p, x_{i+1}\rangle \ldots \langle x_p, x_p\rangle \end{vmatrix} - \sum_{j=1}^{n-p} d_{p+j}\alpha_{ji} \det G_p =$$
$$= \left(c_i - \sum_{j=1}^{n-p} d_{p+j}\alpha_{ji} \right) \det G_p.$$

So, $\sum\limits_{i=1}^{n} d_i \langle x_i, x\rangle = \sum\limits_{i=1}^{p} \left(c_i - \sum\limits_{j=1}^{n-p} d_{p+j}\alpha_{ji} \right) \langle x_i, x\rangle + \sum\limits_{j=1}^{n-p} d_{p+j} \sum\limits_{k=1}^{p} \alpha_{jk} \langle x_k, x\rangle = \sum\limits_{i=1}^{p} c_i \langle x_i, x\rangle.$

Finally, the value of F_n at any stationary point is $\langle x, x\rangle - \sum\limits_{k=1}^{p} c^n_{\tau_n(k)} \left\langle x_{\tau_n(k)}, x\right\rangle$. $\square$

Lemma 2. The stationary points of F_n are points of minimum.

Proof. The function F_n is in fact a second order polynomial with n variables. Writing the Taylor formula at an arbitrary stationary point $(d_1, \ldots, d_n)$ we get (taking into account the fact that the Hessian matrix of F_n is the Gram matrix G_n):

$$F_n(t_1, \ldots, t_n) = F_n(d_1, \ldots, d_n) + \sum_{k=1}^{n} \frac{\partial F_n}{\partial t_k}(d_1, \ldots, d_n)(t_k - d_k) +$$
$$+ Y^T G_n(x_1, \ldots, x_n) Y, \qquad Y = (t_1 - d_1, \ldots, t_n - d_n)^T .$$

Since $Y^T G_n Y \geq 0$ (cf. **P1**, the matrix G_n is positive defined), we obtain:

$$F_n(t_1, \ldots, t_n) \geq F_n(d_1, \ldots, d_n), \quad \forall (t_1, \ldots, t_n) \in \mathbb{R}^n,$$

i.e. $(d_1, \ldots, d_n)$ is a point of minimum. $\square$

The two lemmas show that the minimum value of F_n is $\langle x, x\rangle - \sum\limits_{k=1}^{p} c^n_{\tau_n(k)} \left\langle x_{\tau_n(k)}, x\right\rangle$.

A consequence of lemma 1 is the following equality:

$$\sum_{k=1}^{p} c^n_{\tau_n(k)} \left\langle x_{\tau_n(k)}, x\right\rangle = \sum_{k=1}^{n} c^n_k \langle x_k, x\rangle , \tag{4.1}$$

whence the stated affirmation 2°. $\square$

Relations (4.1) and (3.5) imply:

$$\sum_{k=1}^{n} \langle x_k, x\rangle^2 \geq \lambda_n^{\min}(p) \sum_{k=1}^{n} c_k^n \langle x_k, x\rangle, \ \forall n \in \mathbb{N}^*. \tag{4.2}$$

On the other hand, by **P4** we obtain, using the same equalities as in the consequence 1, section 2, that

$$\sum_{k=1}^{n} \langle x_k, x\rangle^2 \geq \lambda_n^{*\min} \sum_{k=1}^{n} c_k^n \langle x_k, x\rangle, \quad \forall n \in \mathbb{N}^*, \tag{4.3}$$

where $\lambda_n^{*\min}$ is the smallest positive eigenvalue of the matrix G_n.

5 Main result.

Theorem 2. Let $\{x_n, n \in \mathbb{N}^*\}$ be a subset of the separable real Hilbert space $\mathcal{H}$ and G_n the Gram matrices associated to the sets $\{x_k, k = 1, ..., n\}$, $n \in \mathbb{N}^*$. We denote by $\lambda_n^{\min} = \max\left\{\lambda_n^{\min}(p), \lambda_n^{*\min}\right\}$ and $\lambda_n^{\max}$ the largest eigenvalue of G_n.

Let $A = \limsup\limits_{n\to\infty} \lambda_n^{\max}$ and $B = \liminf\limits_{n\to\infty} \lambda_n^{\min}$.

The following statements are true:

1. If $A < \infty$, then $\sum\limits_{k=1}^{\infty} \langle x_k, x\rangle^2 \leq A\|x\|^2$, $\forall x \in \mathcal{H}$.

2. If $\overline{\operatorname{span}\{x_n,\ n \in \mathbb{N}^*\}} = \mathcal{H}$ and $B > 0$, then $B\|x\|^2 \leq \sum\limits_{n=1}^{\infty} \langle x_n, x\rangle^2, \quad \forall x \in \mathcal{H}$.

3. If $A < \infty$, $B > 0$ and $\overline{\operatorname{span}\{x_n,\ n \in \mathbb{N}^*\}} = \mathcal{H}$, then the set $\{x_n, n \in \mathbb{N}^*\}$ forms a frame in $\mathcal{H}$.

Proof.

1. Let $x \in \mathcal{H}$ and $\varepsilon > 0$. There exists $n_0(\varepsilon)$ such that $\lambda_n^{\max} < A+\varepsilon$, $\forall n \geq n_0(\varepsilon)$. From (3.4) if follows that $\sum\limits_{k=1}^{n} \langle x_k, x\rangle^2 \leq (A+\varepsilon) \sum\limits_{k=1}^{n} c_k^n \langle x_k, x\rangle$, $\forall n \geq n_0(\varepsilon)$, where $(c_1^n, ..., c_n^n)$ is an arbitrary solution of (2.1). Take an arbitrary $n \geq n_0(\varepsilon)$. Since $F_n(t_1, ..., t_n) \geq 0$, $\forall (t_1, ..., t_n) \in \mathbb{R}^n$, we have

$$\langle x, x\rangle \geq \sum_{k=1}^{n} c_k^n \langle x_k, x\rangle \geq \frac{1}{A+\varepsilon} \sum_{k=1}^{n} \langle x_k, x\rangle^2.$$

Keeping the extreme sides and passing to limit we get

$$\langle x, x\rangle \geq \frac{1}{A+\varepsilon} \sum_{k=1}^{\infty} \langle x_k, x\rangle^2,$$

inequality which holds for every $\varepsilon > 0$.

So

$$\sum_{k=1}^{\infty} \langle x_k, x\rangle^2 \leq A \|x\|^2 . \tag{5.1}$$

2. Assume that the set $\{x_n, n \in \mathbb{N}^*\}$ is closed in $\mathcal{H}$ ($\overline{\text{span}\{x_n,\ n \in \mathbb{N}^*\}} = \mathcal{H}$), and let $x \in \mathcal{H}$, $\varepsilon > 0$. Then there exist $n(\varepsilon) \in \mathbb{N}^*$ and $(c_k^*)_{k=\overline{1,n(\varepsilon)}}$ such that $F_n(c_1^*, ..., c_n^*) < \varepsilon$, $\forall n \geq n(\varepsilon)$.

From (4.2) and (4.3) it follows that there exists $n_1(\varepsilon)$ such that

$$\sum_{k=1}^{n} \langle x_k, x\rangle^2 \geq (B-\varepsilon) \sum_{k=1}^{n} c_k^n \langle x_k, x\rangle , \quad \forall n \geq n_1(\varepsilon) , \tag{5.2}$$

where $(c_1^n, ..., c_n^n)$ is an arbitrary solution of (2.1). Consider now $n = \max\{n(\varepsilon),\ n_1(\varepsilon)\}$. Taking into account Theorem 1, we have $F_n(c_1^n, ..., c_n^n) \leq F_n(c_1^*, ..., c_n^*) < \varepsilon$, i.e. $\sum\limits_{k=1}^{n} c_k^n \langle x_k, x\rangle > \langle x, x\rangle - \varepsilon$.

Consequently, by (5.2),

$$\sum_{n=1}^{\infty} \langle x_n, x\rangle^2 \geq \sum_{k=1}^{n} \langle x_k, x\rangle^2 \geq (B-\varepsilon) \sum_{k=1}^{n} c_k^n \langle x_k, x\rangle \geq (B-\varepsilon)(\langle x, x\rangle - \varepsilon) .$$

Keeping only the inequality $\sum\limits_{n=1}^{\infty} \langle x_n, x\rangle^2 \geq (B+\varepsilon)(\langle x, x\rangle - \varepsilon)$, which holds for every $\varepsilon > 0$, it follows

$$\sum_{n=1}^{\infty} \langle x_n, x\rangle^2 \geq B \|x\|^2 . \tag{5.3}$$

3. The statement is an immediately consequence of the previous two affirmations. $\square$

6 A particular case.

In the separable real Hilbert space $\mathcal{H}$ we consider the orthonormal set $\{p_n, n \in \mathbb{N}^*\}$. We construct the set $\{e_n, n \in \mathbb{N}^*\}$ in the following way:

$$
(6.1) \qquad \begin{array}{l}
e_1 = a^1_{11} p_1 \\
e_2 = a^1_{21} p_1 + a^1_{22} p_2 \\
\vdots \\
e_k = a^1_{k1} p_1 + ... + a^1_{kk} p_k \\
e_{k+1} = a^2_{11} p_{k+1} \\
e_{k+2} = a^2_{21} p_{k+1} + a^2_{22} p_{k+2} \\
\vdots \\
e_{2k} = a^2_{k1} p_{k+1} + ... + a^2_{kk} p_{2k} \\
\vdots
\end{array},
$$

or $E^1_k = A_1 P^1_k$, $E^{k+1}_k = A_2 P^{k+1}_k, ...,$ where $A_m = \left(a^m_{ij}\right)_{i=\overline{1,k},\ j=\overline{1,i}}$, $E^1_k = (e_1, ..., e_k)^T$, $E^{k+1}_k = (e_{k+1}, ..., e_{2k})^T$,..., $E^l_k = (e_l, ..., e_{l+k-1})^T$, the matrices $P^1_k, P^{k+1}_k, ...$ being defined in an analogous way.

The matrices A_m are triangular and we may assume they have different dimensions. Relations (6.1) which define the set $\{e_n, n \in \mathbb{N}^*\}$ may be written in matrix form: $E = M \cdot P$, where $E = \left(E^1_k E^{k+1}_k ...\right)^T$, $P = \left(P^1_k P^{k+1}_k ...\right)^T$, and M is a block diagonal infinite matrix: $\operatorname{diag}(M) = (A_1 A_2 ...)$.

Denoting by G_n the Gram matrix associated to the blocks $A_1, A_2, ..., A_n$, we get:

$$
\det\left(G_n - \lambda I_{n\cdot k}\right) = \begin{vmatrix} A_1 - \lambda I_k & & & \\ & A_2 - \lambda I_k & & \\ & & \ddots & \\ & & & A_n - \lambda I_k \end{vmatrix}
$$
$$
= \left|A_1 - \lambda I_k\right| \cdot \left|A_2 - \lambda I_k\right| \cdot ... \cdot \left|A_n - \lambda I_k\right|.
$$

In this way, the conditions $A < \infty$ and $B > 0$ are easily satisfied, taking for instance matrices A_m having the same real positive eigenvalues.

The coefficients a^k_{ij} must be taken such that $\overline{\operatorname{span}\{e_n,\ n \in \mathbb{N}^*\}} = \mathcal{H}$.

7 Comparison with existing results.

In [3] is given an equivalent condition under which a frame is a Riesz basis of a separable Hilbert space and there are obtained (using a different approach) formulas for the Riesz bounds: $\limsup\limits_{n\to\infty} \frac{\left(\lambda^{\min}_n\right)^2}{\lambda^{\max}_n}$ for the lower bound B and $\liminf\limits_{n\to\infty} \lambda^{\max}_n$ for the upper bound A. Using **P4** it can be proved that $\frac{\left(\lambda^{\min}_n\right)^2}{\lambda^{\max}_n}$ can be replaced by $\lambda^{\min}_n$ (which is a better value) and, at the same time, an optimal lower bound for the frame $\{x_i : i = \overline{1,n}\}$ considered in [3]. The condition of closedness (condition 2° in Theorem 2) is imposed in [3] too.

The advantage of our results consists in the fact that the set $\{x_i : i \in \mathbb{N}\}$ need not be linearly independent, as it was assumed in [3]. Though, we considered here only real Hilbert spaces.

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