On a possible determination of the frame bounds

Daniela Cătinaş
Cluj-Napoca

1 Introduction.

In a separable Hilbert space $\mathcal{H}$, a subset $\{e_n, n \in \mathbb{N}\}$ is called a frame if there exist $A, B$, $B > 0$, $A < \infty$ (called the frame bounds) such that $B \|x\|^2 \leq \sum_{n \in \mathbb{N}} |\langle x, e_n \rangle|^2 \leq A \|x\|^2$, for every $x \in \mathcal{H}$. For such a sequence, we can find the set $\{\tilde{e}_n, n \in \mathbb{N}\}$ (called the dual frame) having the bounds $A^{-1}, B^{-1}$, and allowing the reconstruction $x = \sum_{n \in \mathbb{N}} \langle x, e_n \rangle \tilde{e}_n = \sum_{n \in \mathbb{N}} \langle x, e_n \rangle e_n$, for every $x \in \mathcal{H}$ (see [2]). The advantage of the frames over the orthonormal and complete bases (which allow the Fourier expansion $x = \sum_{n \in \mathbb{N}} \langle x, e_n \rangle e_n$, $\forall x \in \mathcal{H}$) is that the set $\{e_n, n \in \mathbb{N}\}$ need be neither orthonormal nor linearly independent. Moreover, if $A = B$ (tight frame), than that frame allows the unique expansion $x = A^{-1} \sum_{n \in \mathbb{N}} \langle x, e_n \rangle e_n$, $\forall x \in \mathcal{H}$, similarly to the Fourier one.

We give conditions which ensure that the subset $\{x_n, n \in \mathbb{N}^*\}$ of a separable real Hilbert space $\mathcal{H}$ is a frame, and we obtain formulas for the frame bounds in terms of the eigenvalues of the Gram matrices of the finite subsets.

2 Preliminaries.

In this section we remind some known relations which we shall use in the following.

Let $\{x_n, n \in \mathbb{N}^*\}$ be a subset of the separable real Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, $x \in \mathcal{H}$ and $\langle \cdot, \cdot \rangle_\mathbb{R}$ the standard Euclidean product in $\mathbb{R}^n$. The Gram matrices associated to $\{x_n, n \in \mathbb{N}^*\}$, defined by

$$G_n = G (x_1, \ldots, x_n) = \begin{pmatrix}
\langle x_1, x_1 \rangle & \langle x_1, x_2 \rangle & \cdots & \langle x_1, x_n \rangle \\
\langle x_2, x_1 \rangle & \langle x_2, x_2 \rangle & \cdots & \langle x_2, x_n \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle x_n, x_1 \rangle & \langle x_n, x_2 \rangle & \cdots & \langle x_n, x_n \rangle
\end{pmatrix}, \quad n \in \mathbb{N}^*,$$

have the following properties:
P1: All the eigenvalues of the matrices $G_n$ are nonnegative numbers; if the set \( \{x_1, \ldots, x_n\} \) is linearly independent, then these eigenvalues are positive.

P2: The system

\[
\begin{align*}
    c_1^n \langle x_1, x_1 \rangle &+ c_2^n \langle x_1, x_2 \rangle + \ldots + c_n^n \langle x_1, x_n \rangle = \langle x_1, x \rangle \\
    \vdots & \quad \vdots \\
    c_1^n \langle x_n, x_1 \rangle &+ c_2^n \langle x_n, x_2 \rangle + \ldots + c_n^n \langle x_n, x_n \rangle = \langle x_n, x \rangle
\end{align*}
\]

(2.1)

with the unknowns $c_1^n, c_2^n, \ldots, c_n^n$, is solvable for every $x \in \mathcal{H}$, since if a row of the matrix of coefficients is a linear combination of the other rows, then the same thing happens in the augmented matrix.

If $\text{rank } G_n = p = p(n)$, (for example $x_{\tau_n(1)}, \ldots, x_{\tau_n(p)}$ are linearly independent, where $\tau_n$ is a permutation of the set \( \{1, 2, \ldots, n\} \), and $x_{\tau_n(p+1)}, \ldots, x_{\tau_n(n)}$ are linear combinations of them), then we will consider the solution $(c_1^n, c_2^n, \ldots, c_n^n)$, with $(c_{\tau_n(1)}^n, \ldots, c_{\tau_n(p)}^n)$ as the solution of the linear system

\[
G_{\tau_n(p)} \begin{pmatrix} c_1^n \\ \vdots \\ c_p^n \end{pmatrix} = \begin{pmatrix} \langle x_{\tau_n(1)}, x \rangle \\ \vdots \\ \langle x_{\tau_n(p)}, x \rangle \end{pmatrix},
\]

and $c_{\tau_n(p+1)}^n = \ldots = c_{\tau_n(n)}^n = 0$. (We have denoted $G_{\tau(p)} = G(\tau(1), \ldots, \tau(p))$).

Denoting by $\lambda_n^{\text{min}}$ and $\lambda_n^{\text{max}}$ the smallest, respective the largest eigenvalue of the Gram matrix $G_n$, then the following inequalities hold:

P3: $\lambda_n^{\text{min}} \langle y, y \rangle_e \leq \langle G_n y, y \rangle_e \leq \lambda_n^{\text{max}} \langle y, y \rangle_e$, $\forall y \in \mathbb{R}^n$.

P4: $\lambda_n^{\text{min}} \langle G_n y, y \rangle_e \leq \langle G_n^2 y, y \rangle_e \leq \lambda_n^{\text{max}} \langle G_n y, y \rangle_e$, $\forall y \in \mathbb{R}^n$.

Proof. $\langle G_n y, G_n y \rangle_e - \lambda_n^{\text{max}} \langle G_n y, y \rangle_e = \langle G_n^2 y, y \rangle_e - \langle \lambda_n^{\text{max}} G_n y, y \rangle_e = \langle (G_n^2 - \lambda_n^{\text{max}} G_n) y, y \rangle_e$.

It can be easily proved that if $A$ is a symmetric matrix having the diagonal form $B$, diag $B = (\lambda_n^{\text{min}}, \ldots, \lambda_n^{\text{max}})$, and $P$ is a polynomial, then the matrix $P(A)$ has the diagonal form $C$, with diag $C = (P(\lambda_n^{\text{min}}), \ldots, P(\lambda_n^{\text{max}}))$.

Hence the diagonal form of the matrix $G_n^2 - \lambda_n^{\text{max}} G_n$ is

$$
\begin{pmatrix}
\lambda_n^{\text{min}} & \lambda_n^{\text{min}} - \lambda_n^{\text{max}} \\
\lambda_n^{\text{min}} - \lambda_n^{\text{max}} & \lambda_n^{\text{max}}
\end{pmatrix},
$$

All its eigenvalues are nonpositive numbers, so $\langle (G_n^2 - \lambda_n^{\text{max}} G_n) y, y \rangle_e$ is a negative definite quadratic form, whence the stated inequality.

In the same way, $\langle G_n y, G_n y \rangle_e - \lambda_n^{\text{min}} \langle G_n y, y \rangle_e = \langle (G_n^2 - \lambda_n^{\text{min}} G_n) y, y \rangle_e$. The diagonal form of the matrix $G_n^2 - \lambda_n^{\text{min}} G_n$ is

$$
\begin{pmatrix}
\lambda_n^{\text{min}} & \lambda_n^{\text{min}} - \lambda_n^{\text{min}} \\
\lambda_n^{\text{min}} - \lambda_n^{\text{min}} & \lambda_n^{\text{min}}
\end{pmatrix},
$$

All its eigenvalues are nonnegative numbers, so $\langle (G_n^2 - \lambda_n^{\text{min}} G_n) y, y \rangle_e$ is a positive definite quadratic form, whence the stated inequality.

We will study the set $\{x_n : n \in \mathbb{N}^*\}$, which may be linearly dependent.
3 Properties of the sum $\langle x_1, x \rangle^2 + \ldots + \langle x_n, x \rangle^2$, $x \in \mathcal{H}$, $x \neq 0$.

Consider in the beginning a fixed $n \in \mathbb{N}^*$.

Let $C_n = (c_1^n, c_2^n, \ldots, c_n^n)$ and $X_n = \begin{pmatrix} \langle x_1, x \rangle \\ \vdots \\ \langle x_n, x \rangle \end{pmatrix}$. The system (2.1) may be written as $G_n C_n^T = X_n$.

If we suppose that $x_{\tau_1(n)}, \ldots, x_{\tau_n(p)}$ are linearly independent and $x_{\tau_n(k)}$, $k = p + 1, n$, are linear combinations of them ($p = p(n)$), the solution of the system (2.1), considered in $\mathcal{P}_2$, may be written as:

$$
\begin{cases}
C_p(n) = G_{p(n)}^{-1} X_p(n) \\
c_{\tau_n(p+1)}^n = \ldots = c_{\tau_n(n)}^n = 0,
\end{cases}
$$

where

$$
G_{p(n)} = G (x_{\tau_1(n)}, \ldots, x_{\tau_n(p)}) \quad \text{and} \quad X_p(n) = \begin{pmatrix} \langle x_{\tau_1(n)}, x \rangle \\ \vdots \\ \langle x_{\tau_n(p)}, x \rangle \end{pmatrix}.
$$

Consider the expression

$$
E (X_n) = c_1^n \langle x_1, x \rangle + \ldots + c_n^n \langle x_n, x \rangle = C_n X_n.
$$

In the case when $(c_1^n, \ldots, c_n^n)$ is the solution (3.1) of the system (2.1), it becomes the quadratic form

$$
E (X_p(n)) = C_n X_n = \langle G_{p(n)}^{-1} X_p(n), X_p(n) \rangle_e.
$$

The eigenvalues of the matrix $G_{p(n)}^{-1}$ are the inverses of the eigenvalues of $G_{p(n)}$, (which are positive numbers), so, taking into account $\mathcal{P}_3$, we shall have that:

$$
E (X_p(n)) = \langle G_{p(n)}^{-1} X_p(n), X_p(n) \rangle_e \leq \frac{1}{\lambda_{\min}^{\text{min}} (p)} \langle X_p(n), X_p(n) \rangle_e,
$$

$\lambda_{\min}^{\text{min}} (p)$ being the smallest eigenvalue of the matrix $G_{p(n)}$.

**Remark 1:** Instead of the linearly independent elements $x_{\tau_1(n)}, \ldots, x_{\tau_n(p)}$, we may take other linearly independent elements $x_{\sigma_1(n)}, \ldots, x_{\sigma_n(p)}$ with $\sigma_n$ permutation of $\{1, 2, \ldots, n\}$, different from $\tau_n$. Hence, the matrix $G_{p(n)}$ is not unique, so it is possible to find more values for $\lambda_{\min}^{\text{min}} (p)$. We will choose the largest of them.

**Consequences:**
1. Let \((c_1^n, \ldots, c_n^n)\) be an arbitrary solution of the system (2.1). According to P4 we obtain:

\[
\sum_{k=1}^{n} (x_k, x)^2 = \sum_{k=1}^{n} (c_1^n \langle x_1, x \rangle + \ldots + c_n^n \langle x_n, x \rangle)^2 \\
= \langle G_n C_n^T, G_n C_n^T \rangle_e \leq \lambda_n^{\max} \langle G_n C_n^T, C_n^T \rangle_e \\
= \lambda_n^{\max} \sum_{k=1}^{n} c_k^n (c_1^n \langle x_k, x_1 \rangle + c_2^n \langle x_k, x_2 \rangle + \ldots + c_n^n \langle x_k, x_n \rangle) \\
= \lambda_n^{\max} \sum_{k=1}^{n} c_k^n \langle x_k, x \rangle,
\]

**Remark 2:** When \(p(n) = n\), the above inequality can be immediately obtained: from P3 it follows that

\[
\frac{1}{\lambda_n^{\max}} \langle X_n, X_n \rangle_e \leq \langle G_n^{-1} X_n, X_n \rangle_e = E(X_n), \quad \text{i.e.} \quad \sum_{k=1}^{n} (x_k, x)^2 \leq \lambda_n^{\max} \sum_{k=1}^{n} c_k^n \langle x_k, x \rangle.
\]

2. From (3.3) we obtain:

\[
\langle X_n, X_n \rangle \geq \langle X_{p(n)}, X_{p(n)} \rangle \geq \lambda_n^{\min} (p) E(X_{p(n)}), \quad \text{i.e.}
\]

\[
\sum_{k=1}^{n} (x_k, x)^2 \geq \sum_{k=1}^{p} (x_{\tau_n(k)}, x)^2 \geq \lambda_n^{\min} (p) \sum_{k=1}^{p} c_{\tau_n(k)}^n \langle x_{\tau_n(k)}, x \rangle
\]

where \((c_1^n, \ldots, c_n^n)\) is the solution of the system (2.1) defined by (3.1).

4 **Properties of the sum** \[\sum_{k=1}^{n} c_k \langle x_k, x \rangle, x \in \mathcal{H}, x \neq 0, (c_1, \ldots, c_n)\] **solution of the system (2.1).**

Suppose a fixed \(n \in \mathbb{N}^*\) and \(p\) defined at the beginning of section 2.

**Theorem 1.** Let \(F_n : \mathbb{R}^n \to \mathbb{R}, F_n(t_1, \ldots, t_n) = \langle x - \sum_{k=1}^{n} t_k x_k, x - \sum_{k=1}^{n} t_k x_k \rangle\), and \((c_1^n, \ldots, c_n^n)\) an arbitrary solution of the system (2.1). Then:

1°. If \(p = n\), then \(\min F_n = \langle x, x \rangle - \sum_{k=1}^{n} c_k^n \langle x_k, x \rangle\).

2°. If \(p < n\), then \(\min F_n = \langle x, x \rangle - \sum_{k=1}^{p} c_{\tau_n(k)}^n \langle x_{\tau_n(k)}, x \rangle = \langle x, x \rangle - \sum_{k=1}^{n} c_k^n \langle x_k, x \rangle\).
Proof.

1°. The necessary conditions for extremum, \( \frac{\partial F_n}{\partial x_i} = 0, \ k = \overline{1, n} \), lead to the system (2.1). The Hessian matrix of the function \( F_n \) is, at any point, the Gram matrix \( G(x_1, \ldots, x_n) \), which is strictly positive defined cf. P1. So, \( F_n \) will have a minimum attained at \((c_1^n, \ldots, c_n^n)\), the solution of the system (2.1), namely:

\[
\min F_n = \langle x, x \rangle - \sum_{k=1}^{n} c_k^n \langle x_k, x \rangle.
\]

2°. First we prove the following auxiliary results.

**Lemma 1.** Let \((c_{\tau_1}^n, \ldots, c_{\tau_p}^n)\) be the solution of the system (2.2) and \((d_1^n, \ldots, d_n^n)\) an arbitrary solution of the system (2.1).

Then \( \sum_{k=1}^{n} d_k^n \langle x_k, x \rangle = \sum_{k=1}^{p} c_{\tau(k)}^n \langle x_{\tau(k)}, x \rangle \), i.e. the value of the function \( F_n \) is the same at every stationary point: \( \langle x, x \rangle - \sum_{k=1}^{n} c_{\tau(k)}^n \langle x_{\tau(k)}, x \rangle \).

Proof. For the sake of simplicity we omit the upper indices and we consider that \( x_1, \ldots, x_p \) are linearly independent and \( x_{p+1}, \ldots, x_n \) are linear combinations of them:

\[
x_{p+j} = \sum_{k=1}^{p} \alpha_{jk} x_k, \quad j = \overline{1, n-p}, \quad \text{with } \alpha_{jk} \in \mathbb{R}.
\]

Then the solution of the system (2.2) can be written as \((c_1, \ldots, c_p)\).

The system (2.1) becomes:

\[
\begin{align*}
\langle x_1, x_1 \rangle d_1 + \ldots + \langle x_1, x_p \rangle d_p &= \langle x, x_1 \rangle - \sum_{j=1}^{n-p} \sum_{k=1}^{p} \alpha_{jk} \langle x, x_k \rangle \\
& \vdots \\
\langle x_p, x_1 \rangle d_1 + \ldots + \langle x_p, x_p \rangle d_p &= \langle x, x_p \rangle - \sum_{j=1}^{n-p} \sum_{k=1}^{p} \alpha_{jk} \langle x, x_k \rangle \\
& \vdots \\
\langle x_n, x_1 \rangle d_1 + \ldots + \langle x_n, x_p \rangle d_p &= \langle x, x_n \rangle - \sum_{j=1}^{n-p} \sum_{k=1}^{p} \alpha_{jk} \langle x, x_k \rangle
\end{align*}
\]

Its general solution will be \((d_1, \ldots, d_n)\), with \( d_i = \frac{\det A_i}{\det G_p} \), \( i = \overline{1, p} \) and \( d_{p+1}, \ldots, d_n \) arbitrary, the matrices \( A_i \), \( i = \overline{1, p} \), being given by

\[
A_i = \begin{pmatrix}
\langle x_1, x_1 \rangle \ldots \langle x_1, x_{i-1} \rangle & \langle x, x_1 \rangle - \sum_{j=1}^{n-p} \sum_{k=1}^{p} \alpha_{jk} \langle x, x_k \rangle & \langle x_1, x_{i+1} \rangle \ldots \langle x_1, x_p \rangle \\
& \vdots \\
\langle x_p, x_1 \rangle \ldots \langle x_p, x_{i-1} \rangle & \langle x, x_p \rangle - \sum_{j=1}^{n-p} \sum_{k=1}^{p} \alpha_{jk} \langle x, x_k \rangle & \langle x_p, x_{i+1} \rangle \ldots \langle x_p, x_p \rangle
deprecated
\end{pmatrix}
\]
We get
\[
\sum_{k=1}^{n} d_k \langle x_k, x \rangle = \sum_{i=1}^{p} d_i \langle x_i, x \rangle + \sum_{i=1}^{n-p} d_{p+i} \langle x_{p+i}, x \rangle = \\
= \frac{1}{\det G_p} \sum_{i=1}^{p} (\det A_i) \langle x_i, x \rangle + \sum_{j=1}^{n-p} d_{p+j} \sum_{k=1}^{p} \alpha_{jk} \langle x_k, x \rangle.
\]

If we split \( \det A_i \) after the column \( i \) we obtain:
\[
\det A_i = \begin{vmatrix}
\langle x_1, x_1 \rangle & \langle x_1, x_1 \rangle & \ldots & \langle x_1, x_{i-1} \rangle & \langle x_1, x_{i+1} \rangle & \ldots & \langle x_1, x_p \rangle \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\langle x_p, x_1 \rangle & \langle x_p, x_1 \rangle & \ldots & \langle x_p, x_{i-1} \rangle & \langle x_p, x_{i+1} \rangle & \ldots & \langle x_p, x_p \rangle
\end{vmatrix} - \sum_{j=1}^{n-p} d_{p+j} \alpha_{ji} \det G_p = \\
= \left( c_i - \sum_{j=1}^{n-p} d_{p+j} \alpha_{ji} \right) \det G_p.
\]

So, \( \sum_{i=1}^{n} d_i \langle x_i, x \rangle = \sum_{i=1}^{p} \left( c_i - \sum_{j=1}^{n-p} d_{p+j} \alpha_{ji} \right) \langle x_i, x \rangle + \sum_{j=1}^{n-p} d_{p+j} \sum_{k=1}^{p} \alpha_{jk} \langle x_k, x \rangle = \sum_{i=1}^{p} c_i \langle x_i, x \rangle.
\]

Finally, the value of \( F_n \) at any stationary point is \( \langle x, x \rangle - \sum_{k=1}^{p} c_{\tau_n(k)} \langle x_{\tau_n(k)}, x \rangle \). \( \square \)

**Lemma 2.** The stationary points of \( F_n \) are points of minimum.

**Proof.** The function \( F_n \) is in fact a second order polynomial with \( n \) variables. Writing the Taylor formula at an arbitrary stationary point \( (d_1, \ldots, d_n) \) we get (taking into account the fact that the Hessian matrix of \( F_n \) is the Gram matrix \( G_n \)):
\[
F_n(t_1, \ldots, t_n) = F_n(d_1, \ldots, d_n) + \sum_{k=1}^{n} \frac{\partial F_n}{\partial t_k} (d_1, \ldots, d_n) (t_k - d_k) + \\
y^T G_n (x_1, \ldots, x_n) Y, \quad Y = (t_1 - d_1, \ldots, t_n - d_n)^T.
\]

Since \( Y^T G_n Y \geq 0 \) (cf. P1, the matrix \( G_n \) is positive defined), we obtain:
\[
F_n(t_1, \ldots, t_n) \geq F_n(d_1, \ldots, d_n), \quad \forall (t_1, \ldots, t_n) \in \mathbb{R}^n,
\]
i.e. \( (d_1, \ldots, d_n) \) is a point of minimum. \( \square \)

The two lemmas show that the minimum value of \( F_n \) is \( \langle x, x \rangle - \sum_{k=1}^{p} c_{\tau_n(k)} \langle x_{\tau_n(k)}, x \rangle \).

A consequence of lemma 1 is the following equality:
\[
\sum_{k=1}^{p} c_{\tau_n(k)} \langle x_{\tau_n(k)}, x \rangle = \sum_{k=1}^{n} c_k \langle x_k, x \rangle,
\]
whence the stated affirmation 2°. □

Relations (4.1) and (3.5) imply:

\[
\sum_{k=1}^{n} \langle x_k, x \rangle^2 \geq \lambda_{n}^{\text{min}} (p) \sum_{k=1}^{n} c_k^n \langle x_k, x \rangle, \quad \forall n \in \mathbb{N}^*.
\]  

On the other hand, by P4 we obtain, using the same equalities as in the consequence 1, section 2, that

\[
\sum_{k=1}^{n} \langle x_k, x \rangle^2 \geq \lambda_{n}^{\ast \text{min}} \sum_{k=1}^{n} c_k^n \langle x_k, x \rangle, \quad \forall n \in \mathbb{N}^*,
\]

where \(\lambda_{n}^{\ast \text{min}}\) is the smallest positive eigenvalue of the matrix \(G_n\).

5 Main result.

**Theorem 2.** Let \(\{x_n, n \in \mathbb{N}^*\}\) be a subset of the separable real Hilbert space \(\mathcal{H}\) and \(G_n\) the Gram matrices associated to the sets \(\{x_k, k = 1, \ldots, n\}, n \in \mathbb{N}^*\). We denote by \(\lambda_n^{\text{min}} = \max \{\lambda_n^{\text{min}} (p), \lambda_n^{\ast \text{min}}\}\) and \(\lambda_n^{\text{max}}\) the largest eigenvalue of \(G_n\).

Let \(A = \limsup_{n \to \infty} \lambda_n^{\text{max}}\) and \(B = \liminf_{n \to \infty} \lambda_n^{\text{min}}\).

The following statements are true:

1. If \(A < \infty\), then \(\sum_{k=1}^{\infty} \langle x_k, x \rangle^2 \leq A \|x\|^2, \forall x \in \mathcal{H}\).

2. If \(\overline{\text{span}}\{x_n, n \in \mathbb{N}^*\} = \mathcal{H}\) and \(B > 0\), then \(B \|x\|^2 \leq \sum_{n=1}^{\infty} \langle x_n, x \rangle^2, \quad \forall x \in \mathcal{H}\).

3. If \(A < \infty, B > 0\) and \(\overline{\text{span}}\{x_n, n \in \mathbb{N}^*\} = \mathcal{H}\), then the set \(\{x_n, n \in \mathbb{N}^*\}\) forms a frame in \(\mathcal{H}\).

**Proof.**

1. Let \(x \in \mathcal{H}\) and \(\varepsilon > 0\). There exists \(n_0 (\varepsilon)\) such that \(\lambda_n^{\text{max}} < A + \varepsilon, \forall n \geq n_0 (\varepsilon)\). From (3.4) it follows that \(\sum_{k=1}^{n} \langle x_k, x \rangle^2 \leq (A + \varepsilon) \sum_{k=1}^{n} c_k^n \langle x_k, x \rangle, \forall n \geq n_0 (\varepsilon), \quad \text{where} \ (c_1^n, \ldots, c_n^n) \ \text{is an arbitrary solution of (2.1)}\). Take an arbitrary \(n \geq n_0 (\varepsilon)\). Since \(F_n (t_1, \ldots, t_n) \geq 0, \forall (t_1, \ldots, t_n) \in \mathbb{R}^n\), we have

\[
\langle x, x \rangle \geq \sum_{k=1}^{n} c_k^n \langle x_k, x \rangle \geq \frac{1}{A + \varepsilon} \sum_{k=1}^{n} \langle x_k, x \rangle^2.
\]

Keeping the extreme sides and passing to limit we get

\[
\langle x, x \rangle \geq \frac{1}{A + \varepsilon} \sum_{k=1}^{\infty} \langle x_k, x \rangle^2,
\]
inequality which holds for every $\varepsilon > 0$.

So

$$\sum_{k=1}^{\infty} \langle x_k, x \rangle^2 \leq A \|x\|^2. \tag{5.1}$$

2. Assume that the set $\{x_n, n \in \mathbb{N}^*\}$ is closed in $\mathcal{H}$ (span $\{x_n, n \in \mathbb{N}^*\} = \mathcal{H}$), and let $x \in \mathcal{H}, \varepsilon > 0$. Then there exist $n(\varepsilon) \in \mathbb{N}^*$ and $(c^*_k)_{k=1}^{n(\varepsilon)}$ such that $F_n(c^*_1, \ldots, c^*_n) < \varepsilon, \forall n \geq n(\varepsilon)$.

From (4.2) and (4.3) it follows that there exists $n_1(\varepsilon)$ such that

$$\sum_{k=1}^{n} \langle x_k, x \rangle^2 \geq (B - \varepsilon) \sum_{k=1}^{n} c^n_k \langle x_k, x \rangle, \quad \forall n \geq n_1(\varepsilon), \tag{5.2}$$

where $(c^n_1, \ldots, c^n_n)$ is an arbitrary solution of (2.1). Consider now $n = \max\{n(\varepsilon), n_1(\varepsilon)\}$.

Taking into account Theorem 1, we have $F_n(c^*_1, \ldots, c^*_n) \leq F_n(c^*_1, \ldots, c^*_n) < \varepsilon, i.e.\sum_{k=1}^{n} c^n_k \langle x_k, x \rangle > \langle x, x \rangle - \varepsilon$.

Consequently, by (5.2),

$$\sum_{n=1}^{\infty} \langle x_n, x \rangle^2 \geq \sum_{k=1}^{n} \langle x_k, x \rangle^2 \geq (B - \varepsilon) \sum_{k=1}^{n} c^n_k \langle x_k, x \rangle \geq (B - \varepsilon) (\langle x, x \rangle - \varepsilon).$$

Keeping only the inequality $\sum_{n=1}^{\infty} \langle x_n, x \rangle^2 \geq (B + \varepsilon) (\langle x, x \rangle - \varepsilon)$, which holds for every $\varepsilon > 0$, it follows

$$\sum_{n=1}^{\infty} \langle x_n, x \rangle^2 \geq B \|x\|^2. \tag{5.3}$$

3. The statement is an immediately consequence of the previous two affirmations. \hfill \Box

6 A particular case.

In the separable real Hilbert space $\mathcal{H}$ we consider the orthonormal set $\{p_n, n \in \mathbb{N}^*\}$. We construct the set $\{e_n, n \in \mathbb{N}^*\}$ in the following way:
\[ e_1 = a_{11}^1 p_1 \\
\]
\[ e_2 = a_{12}^1 p_1 + a_{22}^1 p_2 \\
\]
\[ \vdots \\
\]
\[ e_k = a_{k1}^1 p_1 + \ldots + a_{kk}^1 p_k \\
\]
\[ e_{k+1} = a_{11}^{k+1} p_{k+1} \\
\]
\[ e_{k+2} = a_{21}^{k+1} p_{k+1} + a_{22}^{k+2} p_{k+2} \\
\]
\[ \vdots \\
\]
\[ e_{2k} = a_{k1}^{2k} p_{k+1} + \ldots + a_{kk}^{2k} p_{2k} \\
\]

or \( E_k^1 = A_1^1 P_k^1 \), \( E_{k+1}^1 = A_2^1 P_{k+1}^1 \), ..., where \( A_m = (a_{ij}^m)_{i=1,k, \ j=1,k}, \ E_k^1 = (e_1, \ldots, e_k)^T, \ E_{k+1}^1 = (e_{k+1}, \ldots, e_{2k})^T, \ E_k^l = (e_l, \ldots, e_{l+k-1})^T \), the matrices \( P_k^1, P_{k+1}^1, \ldots \) being defined in an analogous way.

The matrices \( A_m \) are triangular and we may assume they have different dimensions. Relations (6.1) which define the set \( \{e_n, n \in \mathbb{N}^*\} \) may be written in matrix form: \( E = M \cdot P \), where \( E = (E_k^1 E_{k+1}^1 \ldots)^T \), \( P = (P_k^1 P_{k+1}^1 \ldots)^T \), and \( M \) is a block diagonal infinite matrix: \( \text{diag}(M) = (A_1 A_2 \ldots) \).

Denoting by \( G_n \) the Gram matrix associated to the blocks \( A_1, A_2, \ldots, A_n \), we get:

\[
\det (G_n - \lambda I_n) = \begin{vmatrix}
A_1 - \lambda I_k & A_2 - \lambda I_k & \ldots & A_n - \lambda I_k \\
A_2 - \lambda I_k & A_2 - \lambda I_k & \ldots & A_n - \lambda I_k \\
\vdots & \vdots & \ddots & \vdots \\
A_n - \lambda I_k & A_n - \lambda I_k & \ldots & A_n - \lambda I_k
\end{vmatrix} = |A_1 - \lambda I_k| \cdot |A_2 - \lambda I_k| \cdot \ldots \cdot |A_n - \lambda I_k|.
\]

In this way, the conditions \( A < \infty \) and \( B > 0 \) are easily satisfied, taking for instance matrices \( A_m \) having the same real positive eigenvalues.

The coefficients \( a_{ij}^k \) must be taken such that \( \text{span} \{e_n, n \in \mathbb{N}^*\} = \mathcal{H} \).

### 7 Comparison with existing results.

In [3] is given an equivalent condition under which a frame is a Riesz basis of a separable Hilbert space and there are obtained (using a different approach) formulas for the Riesz bounds: \( \limsup_{n \to \infty} \frac{(\lambda_{n}^{\min})^2}{\lambda_{n}^{\max}} \) for the lower bound \( B \) and \( \liminf_{n \to \infty} \lambda_{n}^{\max} \) for the upper bound \( A \).

Using \( \textbf{P4} \) it can be proved that \( \frac{(\lambda_{n}^{\min})^2}{\lambda_{n}^{\max}} \) can be replaced by \( \lambda_{n}^{\min} \) (which is a better value)
and, at the same time, an optimal lower bound for the frame \( \{x_i : i = 1, n\} \) considered in [3]. The condition of closedness (condition 2° in Theorem 2) is imposed in [3] too.

The advantage of our results consists in the fact that the set \( \{x_i : i \in \mathbb{N}\} \) need not be linearly independent, as it was assumed in [3]. Though, we considered here only real Hilbert spaces.

References


Daniela Cătinaș
Technical University of Cluj-Napoca
Department of Mathematics
str. Gh. Baritiu 25
3400 Cluj-Napoca
Romania
e-mail: Daniela.Catinas@math.utcluj.ro