A Class of Combinatorial Identities

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In [1] some combinatorial identities were used in order to prove that some matrices from the wavelet theory are triangularizable. Since these identities are interesting by their own, we present them here, together with their proof.

Proposition 1 *Let $p \in \mathbb{N}^*$. Then the following identities hold:*

$$\sum_{m=0}^{p} \binom{2p}{2m}\binom{k+m-1}{2p-1} = \binom{2k-1}{2p-1}, \quad k = 2p,\ 2p+1, \ldots \tag{1}$$

$$\sum_{m=1}^{p} \binom{2p}{2m-1}\binom{k+m-2}{2p-1} = \binom{2k-2}{2p-1}, \quad k = 2p,\ 2p+1, \ldots \tag{2}$$

$$\sum_{m=0}^{p} \binom{2p+1}{2m}\binom{k+m-1}{2p} = \binom{2k-1}{2p}, \quad k = 2p+1,\ 2p+2, \ldots \tag{3}$$

$$\sum_{m=1}^{p+1} \binom{2p+1}{2m-1}\binom{k+m-2}{2p} = \binom{2k-2}{2p}, \quad k = 2p+1,\ 2p+2, \ldots \tag{4}$$

Proof. We will prove only the first identity, since for the others the proofs work analogously. For this we show that the polynomials

$$\begin{aligned} P(x) &= \sum_{m=0}^{p} \binom{2p}{2m} (x+m-1)(x+m-2)\ldots(x+m-(2p-1)) \text{ and} \\ Q(x) &= (2x-1)(2x-2)\ldots(2x-(2p-1)), \end{aligned}$$

of degree $2p-1$, coincide. Then they will coincide at the points $k = 2p,\ 2p+1, \ldots$, fact which proves the identity.

In order to prove that $P = Q$, we see first that the coefficient of x^{2p-1} in both P and Q is 2^{2p-1}. Then we prove that they have the same roots.

It is immediately that $P(j) = Q(j) = 0$ for $j = 1, 2, \ldots, p-1$. So, it remains to prove that $P\left(\frac{2s-1}{2}\right) = 0$ for $s = 1, 2, \ldots, p$. Denoting

$$R(x,t) = \sum_{m=0}^{p} \binom{2p}{2m} (x+m-1)(x+m-2)\ldots(x+m-t), \text{ for } t \in \mathbb{N}^*, \tag{5}$$

we may write $R(x, 2p-1) = P(x)$.

Thus, we have to prove that $R\left(\frac{2s-1}{2}, 2p-1\right) = 0$, for $s = 1, 2, \ldots, p$. For this, we write a recurrence formula as follows. First we easily deduce that

$$R(x,t) = R(x-1,t) + tR(x-1,t-1).$$

Then, by induction on n we immediately obtain

$$\begin{aligned} R(x,t) &= R(x-n,t) + \tbinom{n}{1} tR(x-n,t-1) + \tbinom{n}{2} t(t-1)R(x-n,t-2) + \ldots \\ &+ \tbinom{n}{i} t(t-1)\ldots(t-i+1)R(x-n,t-i) + \ldots \\ &+ \tbinom{n}{n} t(t-1)\ldots(t-n+1)R(x-n,t-n), \end{aligned} \tag{6}$$

for $1 \leq n < \min(x,t)$.

So $R(x,t)$ is combination of $R(x-n,t)$, $R(x-n,t-1), \ldots, R(x-n,t-n)$. Writing the relation (6) for $t = 2p-1$, $x = (2s-1)/2$, $s = 2, 3, \ldots, p$ and $n = s-1$, we get

$$\begin{aligned} R\left(\frac{2s-1}{2}, 2p-1\right) &= R\left(\frac{1}{2}, 2p-1\right) + \tbinom{s-1}{1}(2p-1)R\left(\frac{1}{2}, 2p-2\right) + \\ &+ \tbinom{s-1}{2}(2p-1)(2p-2)R\left(\frac{1}{2}, 2p-3\right) + \ldots \\ &+ \tbinom{s-1}{s-1}(2p-1)\ldots(2p-s+1)R\left(\frac{1}{2}, 2p-s\right), \end{aligned} \tag{7}$$

for $s = 2, \ldots, p$.

In order to have $R\left(\frac{2s-1}{2}, 2p-1\right) = 0$ for $s = 1, \ldots, p$, it is enough to prove that

$$R\left(\frac{1}{2}, p+t\right) = 0 \text{ for } t = 0, 1, \ldots, p-1.$$

Evaluating $R\left(\frac{1}{2}, p\right)$ we obtain, using formula (6),

$$\begin{aligned} R\left(\frac{1}{2}, p\right) &= \sum_{m=0}^{p} \tbinom{2p}{2m} \left(\frac{1}{2} + m - 1\right) \ldots \left(\frac{1}{2} + m - p\right) \\ &= \frac{1}{2^p} \sum_{m=0}^{p} \tbinom{2p}{2m} (2m-1)(2m-3)\ldots(2m-2p+1). \end{aligned}$$

Consider now the polynomial

$$H(x) = \sum_{m=0}^{p} \binom{2p}{2m} (2m-1)(2m-3)\ldots(2m-2p+1)\, x^m.$$

A simple induction on m shows that

$$\frac{\binom{2p}{2m}(2m-1)\dots(2m-2p+1)}{(-1)^p(2p-1)!!} = (-1)^m \binom{p}{m}.$$

Thus, the polynomial H gets the expression

$$H(x) = \frac{1}{(-1)^p(2p-1)!!}\sum_{m=0}^{p}(-1)^m\binom{p}{m}x^m = \frac{(-1)^p}{(2p-1)!!}(1-x)^p,$$

whence $H(1) = H'(1) = \dots = H^{(p-1)}(1) = 0$.

In conclusion we have $0 = H(1) = 2^p R\left(\frac{1}{2}, p\right)$ and further, for $t = 1, 2, \dots, p-1$,

$$\begin{aligned} H^{(t)}(1) &= \sum_{m=t}^{p}\binom{2p}{2m}(2m-1)(2m-3)\dots(2m-2p+1)\,m(m-1)\dots(m-t+1) \\ &= \sum_{m=0}^{p}\binom{2p}{2m}(2m-1)(2m-3)\dots(2m-2p+1)\,m(m-1)\dots(m-t+1) \\ &= 0. \end{aligned}$$

Now, evaluating $R\left(\frac{1}{2}, p+t\right)$ from (5), it follows that

$$\begin{aligned} R\left(\frac{1}{2}, p+t\right) &= \frac{1}{2^{p+t}}\sum_{m=0}^{p}\binom{2p}{2m}(2m-1)(2m-3)\dots(2m-2p+1)\,\cdot \\ &\quad \cdot\left[(2m-2p-1)\dots(2m-2p-2t+1)\right] \\ &= \frac{1}{2^{p+t}}\sum_{m=0}^{p}\binom{2p}{2m}(2m-1)(2m-3)\dots(2m-2p+1)\left[a_0 + \right. \\ &\quad \left. a_1 m + a_2 m(m-1) + \dots a_t m(m-1)\dots(m-t+1)\right] \\ &= \frac{1}{2^{p+t}}\left(a_0 H(1) + a_1 H'(1) + \dots + a_t H^{(t)}(1)\right) \\ &= 0, \end{aligned}$$

for $t = 1, \dots, p-1$. Using now (7), yields

$$R\left(\frac{2s-1}{2}, 2p-1\right) = 0 \text{ for } s = 1, \dots, p.$$

In conclusion the polynomials P and Q coincide, fact which proves the relation (1). Analogously we can prove the relations (2), (3) and (4). ■

References

[1] D. Roşca, *Some sufficient conditions for the convergence of the cascade algorithm and for the continuity of the scaling function,* submitted.