Optimal Haar Wavelets on Spherical Triangulations[1]

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Abstract

In a previous paper we constructed some Haar wavelets on spherical triangulations, which are orthogonal with respect to a weighted inner product on $L^2(\mathbb{S}^2)$. We obtained two classes of wavelets which included certain wavelets obtained by Bonneau and Nielson et al. Each of these classes depended on two parameters which satisfied a relation. In this paper we study which of these wavelets are optimal with respect to two norms of the compression error.

1 Introduction

For the beginning we remind the construction of piecewise constant wavelets made in [4]. These wavelets are orthogonal with respect to a weighted inner product on $L^2(\mathbb{S}^2)$, introduced in [5]. An extension of this construction to closed surfaces is presented in [6].
Let $\mathbb{S}^2$ be the unit sphere of $\mathbb{R}^3$ with center O and radius 1 and Π a convex polyhedron having triangular faces and the vertices situated on the sphere $\mathbb{S}^2$.The polyhedron could also have faces which are not triangles. In that case we triangulate each of these faces and consider it as having triangular faces, with some of the faces coplanar. Also we have to suppose that no face contains the origin O and O is situated inside the polyhedron. We denote by $\mathcal{T}^0 = \{T_1, T_2, \ldots, T_n\}$ the set of the faces of Π and by Ω the surface (the "cover") of Π. Then we consider the radial projection onto $\mathbb{S}^2$, $p : \Omega \to \mathbb{S}^2$,

$$p\,(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}\,(x, y, z)\,,\ (x, y, z) \in \Omega.$$

[1]This is an erata of the paper published in Pure Math Appl. 15, 2-3 (2005), pp. 295-302.

Being given Ω, we can say that $\mathcal{T} = \mathcal{T}^0$ is a triangulation of Ω. Next we consider its uniform refinement $\mathcal{T}^1$. For a given triangle $[M_1M_2M_3]$ in $\mathcal{T}^0$, let A_1, A_2, A_3 denote the midpoints of the edges M_2M_3, M_3M_1 and M_1M_2, respectively. Then we consider the set

$$\mathcal{T}^1 = \bigcup_{[M_1M_2M_3]\in\mathcal{T}^0} \{[M_1A_2A_3], [A_1M_2A_3], [A_1A_2M_3], [A_1A_2A_3]\},$$

which is also a triangulation of Ω. Continuing in the same manner the refinement process we can obtain a triangulation $\mathcal{T}^j$ of Ω, for $j \in \mathbb{N}$. The projection of $\mathcal{T}^j$ onto the sphere will be $\mathcal{U}^j = \{p(T^j),\ T^j \in \mathcal{T}^j\}$, which is a triangulation of $\mathbb{S}^2$. The number of triangles in $\mathcal{U}^j$ will be $|\mathcal{U}^j| = n \cdot 4^j$.

Let $\langle\cdot,\cdot\rangle_\Omega$ be the following inner product, based on the initial coarsest triangulation $\mathcal{T}^0$:

$$\langle f, g\rangle_\Omega = \sum_{T\in\mathcal{T}^0} \frac{1}{a(T)} \int_T f(\mathbf{x})\, g(\mathbf{x})\, d\mathbf{x}, \text{ for } f, g \in C(T)\ \ \forall T \in \mathcal{T}^0.$$

Here $a(T)$ denotes the area of the triangle T. Also, we consider the induced norm

$$\|f\|_\Omega = \langle f, f\rangle_\Omega^{1/2}.$$

For the L^2-integrable functions F and G defined on $\mathbb{S}^2$, the following inner product associated to the given polyhedron Π was defined in [5]:

$$\langle F, G\rangle_{*,\mathbb{S}^2} = \langle F\circ p, G\circ p\rangle_\Omega. \tag{1}$$

There it was proved that, in the space $L^2(\mathbb{S}^2)$, the norm $\|\cdot\|_{*,\mathbb{S}^2}$ induced by this inner product is equivalent to the usual norm $\|\cdot\|_{L^2(\mathbb{S}^2)}$ of $L^2(\mathbb{S}^2)$.
Then we constructed a multiresolution on $\mathbb{S}^2$ consisting of piecewise constant functions on the triangles of $\mathcal{U}^j = \left\{U_1^j, U_2^j, \ldots, U_{n\cdot 4^j}^j\right\}$, $j \in \mathbb{N}$.

By definition, a multiresolution of $L^2(\mathbb{S}^2)$ is a sequence of subspaces $\{\mathcal{V}^j : j \geq 0\}$ of $L^2(\mathbb{S}^2)$ which satisfies the following properties:

1. $\mathcal{V}^j \subseteq \mathcal{V}^{j+1}$ for all $j \in \mathbb{N}$,
2. $\operatorname{clos}_{L^2(\mathbb{S}^2)} \bigcup_{j=0}^{\infty} \mathcal{V}^j = L^2(\mathbb{S}^2)$,
3. There are index sets $\mathcal{K}_j \subseteq \mathcal{K}_{j+1}$ such that for every level j there exists a Riezs basis $\left\{\varphi_t^j,\ t \in \mathcal{K}_j\right\}$ of the space $\mathcal{V}^j$. This means that there exist constants $0 < c \leq C < \infty$, independent of the level j, such that

$$c2^{-j}\left\|\left\{c_t^j\right\}_{t\in\mathcal{K}^j}\right\|_{l_2(\mathcal{K}_j)} \leq \left\|\sum_{t\in\mathcal{K}^j} c_t^j\varphi_t^j\right\|_{L^2(\mathbb{S}^2)} \leq C2^{-j}\left\|\left\{c_t^j\right\}_{t\in\mathcal{K}^j}\right\|_{l_2(\mathcal{K}_j)}.$$

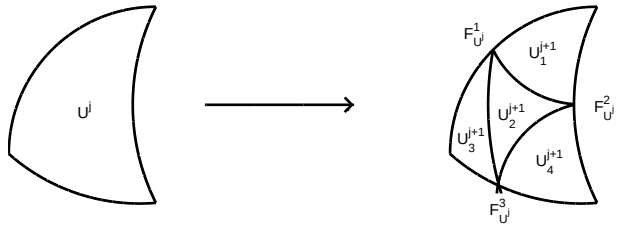

Figure 1: The triangle U^j and its refined triangles $U_k^{j+1},\ k=1,2,3,4$.

For a fixed $j \in \mathbb{N}$, to each triangle $U_k^j \in \mathcal{U}^j,\ k = 1,2,\ldots,n\cdot 4^j$, we associated the function $\varphi_{U_k^j} : \mathbb{S}^2 \to \mathbb{R}$,

$$\varphi_{U_k^j}(\eta) = \begin{cases} 1, & \text{inside the triangle } U_k^j\,, \\ 1/2, & \text{on the edges of } U_k^j\,, \\ 0, & \text{elsewhere.} \end{cases}$$

Then we constructed the spaces of functions

$$\mathcal{V}^j = \operatorname{span}\left\{\varphi_{U_k^j},\ k = 1,2,\ldots,n\cdot 4^j\right\},$$

consisting of piecewise constant functions on the triangles of $\mathcal{U}^j$. If $U_k^{j+1} = p\left(T_k^{j+1}\right),\ k=1,2,3,4$ are the refined triangles obtained from U^j as in Figure 1, we have

$$\varphi_{U^j} = \varphi_{U_1^{j+1}} + \varphi_{U_2^{j+1}} + \varphi_{U_3^{j+1}} + \varphi_{U_4^{j+1}},$$

which holds in $L^2(\mathbb{S}^2)$. Thus, $\mathcal{V}^j \subseteq \mathcal{V}^{j+1}$ for all $j \in \mathbb{N}$. With respect to the scalar product $\langle\cdot,\cdot\rangle_{*,\mathbb{S}^2}$, the spaces $\mathcal{V}^j$ and $\mathcal{V}^{j+1}$ become Hilbert spaces, with the corresponding norm $\|\cdot\|_{*,\mathbb{S}^2} = \langle\cdot,\cdot\rangle_{*,\mathbb{S}^2}^{1/2}$.

Next we defined the space $\mathcal{W}^j$ as the orthogonal complement, with respect to the scalar product $\langle\cdot,\cdot\rangle_{*,\mathbb{S}^2}$, of the coarse space $\mathcal{V}^j$ in the fine space $\mathcal{V}^{j+1}$:

$$\mathcal{V}^{j+1} = \mathcal{V}^j \bigoplus \mathcal{W}^j.$$

The spaces $\mathcal{W}^j$ are called *the wavelet spaces.* The dimension of $\mathcal{W}^j$ is $|\mathcal{W}^j| = |\mathcal{V}^{j+1}| - |\mathcal{V}^j| = 3n\cdot 4^j$. In [4] we proved that we have two classes of orthonormal[2] wavelet bases. With the notations of Figure 1, these wavelets

[2]The orthogonality is with respect to the norm $\|\cdot\|_{*,\mathbb{S}^2}$

have the expressions

$$
\begin{aligned}
{}^1\Psi_{F^1_{j+1},U^j} &= \alpha_1\varphi_{U^{j+1}_1} + \alpha_2\varphi_{U^{j+1}_3} + \frac{1}{2}\varphi_{U^{j+1}_2} - \left(\frac{1}{2} + \alpha_1 + \alpha_2\right)\varphi_{U^{j+1}_4},\\
{}^1\Psi_{F^2_{j+1},U^j} &= \alpha_1\varphi_{U^{j+1}_4} + \alpha_2\varphi_{U^{j+1}_1} + \frac{1}{2}\varphi_{U^{j+1}_2} - \left(\frac{1}{2} + \alpha_1 + \alpha_2\right)\varphi_{U^{j+1}_3},\\
{}^1\Psi_{F^3_{j+1},U^j} &= \alpha_1\varphi_{U^{j+1}_3} + \alpha_2\varphi_{U^{j+1}_4} + \frac{1}{2}\varphi_{U^{j+1}_2} - \left(\frac{1}{2} + \alpha_1 + \alpha_2\right)\varphi_{U^{j+1}_1},
\end{aligned}
$$

with (α_1, α_2) situated on the ellipse E_1 of equation $4\left(\alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2\right) + 2\left(\alpha_1 + \alpha_2\right) - 1 = 0$
and

$$
\begin{aligned}
{}^2\Psi_{F^1_{j+1},U^j} &= \alpha_1\varphi_{U^{j+1}_1} + \alpha_2\varphi_{U^{j+1}_3} - \frac{1}{2}\varphi_{U^{j+1}_2} + \left(\frac{1}{2} - \alpha_1 - \alpha_2\right)\varphi_{U^{j+1}_4},\\
{}^2\Psi_{F^2_{j+1},U^j} &= \alpha_1\varphi_{U^{j+1}_4} + \alpha_2\varphi_{U^{j+1}_1} - \frac{1}{2}\varphi_{U^{j+1}_2} + \left(\frac{1}{2} - \alpha_1 - \alpha_2\right)\varphi_{U^{j+1}_3},\\
{}^2\Psi_{F^3_{j+1},U^j} &= \alpha_1\varphi_{U^{j+1}_3} + \alpha_2\varphi_{U^{j+1}_4} - \frac{1}{2}\varphi_{U^{j+1}_2} + \left(\frac{1}{2} - \alpha_1 - \alpha_2\right)\varphi_{U^{j+1}_1},
\end{aligned}
$$

with (α_1, α_2) situated on the ellipse E_2 of equation $4\left(\alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2\right) - 2\left(\alpha_1 + \alpha_2\right) - 1 = 0$.
Let us mention that, for some particular values of (α_1, α_2), one obtains the wavelets constructed by Bonneau in [1] and by Nielson et al. in [2].
One of the most important application of wavelets is data compression. The idea is to decompose a sequence s with respect to the wavelet basis, to cancel those wavelet coefficients $(w_k)_k$ which are small and then to reconstruct the sequence s using the left coefficients. The result is an approximation $\widehat{s}$ of s. The advantage is that for storing into the memory the sequence $\widehat{s}$, less memory is needed. This is because the place of some coefficients $(w_k)_k$ is taken by zeros.
The goal of this paper is to find those numbers (α_1, α_2) for which some errors of the vector $s - \widehat{s}$ are minimized.

2 Decomposition, compression, reconstruction

Let $i \in \mathbb{N}^*$ be a given level. For our case, suppose that the sequence $(f^i_k)_{k=1,\ldots,n\cdot 4^i}$ is given and it represents the values of the function φ^i, which is piecewise constant on the spherical triangles at the level i,

$$\varphi^i = \sum_{U\in\mathcal{U}^i} f^i_k \cdot \varphi^i_U.$$

`Decomposition`

Given the column vector $f^i = (f^i_k)_{k=1,\dots,n\cdot 4^i}$, we calculate the column vectors $w^{i-1}, w^{i-2}, \dots, w^0, f^0$ in the following way.

$$\begin{pmatrix} f^{j-1} \\ w^{j-1} \end{pmatrix} = W_j^T \cdot f^j,$$

for $j = i, i-1, \dots, 1$, where f^{j-1} and w^{j-1} are column vectors which have the dimension $n \cdot 4^{j-1}$ and $3n \cdot 4^{j-1}$ respectively, and W_j is the block matrix

$$W_j = \left(\begin{array}{c|c} P_j & Q_j \end{array} \right) = \left(\begin{array}{cccc|cccc} u & & & & m & & & \\ & u & & & & m & & \\ & & \ddots & & & & \ddots & \\ & & & u & & & & m \end{array} \right),$$

with

$$u = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}, \quad m = \begin{pmatrix} \alpha_1 & \alpha_2 & \gamma \\ \beta & \beta & \beta \\ \alpha_2 & \gamma & \alpha_1 \\ \gamma & \alpha_1 & \alpha_2 \end{pmatrix}.$$

Here $\gamma = -0.5 - \alpha_1 - \alpha_2,\ \beta = 0.5$ for the wavelets ${}^1\Psi$ and $\gamma = 0.5 - \alpha_1 - \alpha_2,\ \beta = -0.5$ for the wavelets ${}^2\Psi$. Thus, the matrix P_j has $n \cdot 4^j$ rows and $n \cdot 4^{j-1}$ columns, while the matrix Q_j has $n \cdot 4^j$ rows and $3n \cdot 4^{j-1}$ columns. Due to the orthogonality properties of our wavelets, the matrix W_j is orthogonal, meaning that $W_j \cdot W_j^T = W_j^T \cdot W_j = I_{n \cdot 4^j}$.

The components of the vectors $w^i, w^{i-1}, \dots, w^0$ are the called the wavelet coefficients at the levels $i, i-1, \dots, 0$.

The decomposition scheme is

$$\begin{array}{ccccccccc} & & w^{i-1} & & w^{i-2} & & \cdots & & w^0 \\ & \nearrow & & \nearrow & & \nearrow & & \nearrow & \\ f^i & \longrightarrow & f^{i-1} & \longrightarrow & f^{i-2} & \longrightarrow & \cdots & \longrightarrow & f^0. \end{array}$$

`Compression`

The idea of compression is to cancel those wavelet coefficients which are small. A very used procedure is to cancel the smallest r ($r \in \mathbb{N}$ given number) coefficients. The result is the vectors $\widehat{w}^i, \widehat{w}^{i-1}, \dots, \widehat{w}^0$, having the components

$$\widehat{w}^j_k = \begin{cases} 0, & \text{if } w^j_k \text{ is among the smallest } r \text{ coefficients,} \\ w^j_k, & \text{otherwise,} \end{cases}$$

$j = 0, \dots, i-1,\ k = 1, \dots, 3n \cdot 4^{j-1}$.

`Reconstruction`

With the help of the compressed wavelet coefficients in $\widehat{w}^j,\ j=0,1,\dots,i-1$, we can obtain an approximated version $\widehat{f^i}$ of the initial vector f^i as follows.

$$\widehat{f^j} = W_j \cdot \begin{pmatrix} \widehat{f^{j-1}} \\ \widehat{w}^{j-1} \end{pmatrix}, \tag{2}$$

for $j=1,2,\dots,i$ and with the notation $\widehat{f^0}=f^0$.
The reconstruction scheme is

$$\begin{array}{ccccccccc} \widehat{w}^0 & & \widehat{w}^1 & & \widehat{w}^2 & & \cdots & \widehat{w}^{i-1} & \\ & \searrow & & \searrow & & \searrow & & & \searrow \\ \widehat{f^0} & \longrightarrow & \widehat{f_1} & \longrightarrow & \widehat{f_2} & \longrightarrow & \cdots & \widehat{f^{i-1}} & \longrightarrow & \widehat{f^i}. \end{array}$$

2.1 The compression error

The error of the compression can be measured by a norm of the vector $f^i-\widehat{f^i}$. In the following we consider the norms $\|\cdot\|_2$, and $\|\cdot\|_\infty$ defined by

$$\|f^i-\widehat{f^i}\|_2 = \left(\sum_{k=1}^{n\cdot 4^i}\left(f^i_k-\widehat{f^i_k}\right)^2\right)^{1/2},$$

$$\|f^i-\widehat{f^i}\|_\infty = \max\left\{|f^i_k-\widehat{f^i_k}|,\ k\in\{1,\dots,n\cdot 4^i\}\right\}$$

and try to find out for which (α_1,α_2) (which of the wavelets) this norms are minim, in the case when $\widehat{f^i}$ is obtained from f^i by making zero one fixed wavelet coefficient $w^{i_0}_{k_0}$.
For all $j\in\{i,i-1,\dots,1\}$, in the vector f^j having $n\cdot 4^j$ components, we make groups of four components and denote them by $\mathbf{f}^j_1,\mathbf{f}^j_2,\dots,\mathbf{f}^j_{n\cdot 4^{j-1}}$. More precisely, $\mathbf{f}^j_k$ will be the vector

$$\mathbf{f}^j_k = \begin{pmatrix} f^j_{4k-3} & f^j_{4k-2} & f^j_{4k-1} & f^j_{4k} \end{pmatrix}^T,\ k=1,2,\dots,n\cdot 4^{j-1}.$$

Therefore we can write

$$W_j^T\cdot f^j = \left(\ u^T\mathbf{f}^j_1 \mid u^T\mathbf{f}^j_2 \mid \cdots \mid u^T\mathbf{f}^j_{n\cdot 4^{j-1}} \mid m^T\mathbf{f}^j_1 \mid m^T\mathbf{f}^j_2 \mid \cdots \mid m^T\mathbf{f}^j_{n\cdot 4^{j-1}}\ \right)^T.$$

Thus, the wavelet coefficients are the components of the vectors $m^T\mathbf{f}^j_k$, for $j=i,i-1,\dots,1$ and $k=1,2,\dots,3n\cdot 4^{j-1}$, as follows.

$$m^T\mathbf{f}^j_k = \begin{pmatrix} w^{j-1}_{3k-2} \\ w^{j-1}_{3k-1} \\ w^{j-1}_{3k} \end{pmatrix} = \mathbf{w}^j_k,\ \text{for } k=1,\dots,n\cdot 4^{j-1}.$$

Suppose that $w_{k_0}^{i_0}$ is the wavelet coefficient which is going to be cancelled. There is a unique $p \in \mathbb{N}^*$ such that $k_0 = 3p - l$, with $l \in \{0, 1, 2\}$ and we have

$$m^T \mathbf{f}_p^{i_0+1} = \begin{pmatrix} w_{3p-2}^{i_0} \\ w_{3p-1}^{i_0} \\ w_{3p}^{i_0} \end{pmatrix} = \mathbf{w}_p^{i_0}. \tag{3}$$

Let $r = 3 - l$. Replacing $w_{k_0}^{i_0}$ by zero means that we replace by zero the r-th component of the vector in (3), obtaining a vector denoted by $\widehat{\mathbf{w}}_p^{i_0}$, which depends on k_0. Thus, applying the relation (2) successively for $j = i_0 + 1, i_0 + 2, ..., i$ we obtain

$$\widehat{f}^{i_0+1} = \begin{pmatrix} uf_1^{i_0} + m\mathbf{w}_1^{i_0} \\ \cdots \\ uf_{p-1}^{i_0} + m\mathbf{w}_{p-1}^{i_0} \\ uf_p^{i_0} + m\widehat{\mathbf{w}}_p^{i_0} \\ uf_{p+1}^{i_0} + m\mathbf{w}_{p+1}^{i_0} \\ \cdots \\ uf_{n\cdot 4^{i_0}}^{i_0} + m\mathbf{w}_{n\cdot 4^{i_0}}^{i_0} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1^{i_0+1} \\ \cdots \\ \mathbf{f}_{p-1}^{i_0+1} \\ \widehat{\mathbf{f}}_p^{i_0+1} \\ \mathbf{f}_{p+1}^{i_0+1} \\ \cdots \\ \mathbf{f}_{n\cdot 4^{i_0}}^{i_0+1} \end{pmatrix}, \; f^{i_0+1} = \begin{pmatrix} uf_1^{i_0} + m\mathbf{w}_1^{i_0} \\ \cdots \\ uf_p^{i_0} + m\mathbf{w}_p^{i_0} \\ \cdots \\ uf_{n\cdot 4^{i_0}}^{i_0} + m\mathbf{w}_{n\cdot 4^{i_0}}^{i_0} \end{pmatrix},$$

whence

$$\left\| f^{i_0+1} - \widehat{f}^{i_0+1} \right\| = \left\| \mathbf{f}_p^{i_0+1} - \widehat{\mathbf{f}}_p^{i_0+1} \right\| = \left\| m \left(\mathbf{w}_p^{i_0} - \widehat{\mathbf{w}}_p^{i_0} \right) \right\| = |w_{k_0}^{i_0}| \cdot \|m_r\|,$$

where m_r denotes the column r of the matrix m and $\| \cdot \|$ is an arbitrary vector norm. Then

$$\begin{aligned} \widehat{f}^{i_0+2} &= \left(\; P_{i_0+2} \mid Q_{i_0+2} \; \right) \begin{pmatrix} \widehat{f}^{i_0+1} \\ \widehat{w}^{i_0+1} \end{pmatrix} = \left(\; P_{i_0+2} \mid Q_{i_0+2} \; \right) \begin{pmatrix} \widehat{f}^{i_0+1} \\ w^{i_0+1} \end{pmatrix} \\ &= \begin{pmatrix} uf_1^{i_0+1} + m\mathbf{w}_1^{i_0+1} \\ \cdots \\ uf_{4p-4}^{i_0+1} + m\mathbf{w}_{4p-4}^{i_0+1} \\ u\widehat{f}_{4p-3}^{i_0+1} + m\mathbf{w}_{4p-3}^{i_0+1} \\ \cdots \\ u\widehat{f}_{4p}^{i_0+1} + m\mathbf{w}_{4p}^{i_0+1} \\ uf_{4p+1}^{i_0+1} + m\mathbf{w}_{4p+1}^{i_0+1} \\ \cdots \\ uf_{n\cdot 4^{i_0+1}}^{i_0+1} + m\mathbf{w}_{n\cdot 4^{i_0+1}}^{i_0+1} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1^{i_0+2} \\ \cdots \\ \mathbf{f}_{4p-4}^{i_0+2} \\ \widehat{\mathbf{f}}_{4p-3}^{i_0+2} \\ \cdots \\ \widehat{\mathbf{f}}_{4p}^{i_0+2} \\ \mathbf{f}_{4p+1}^{i_0+2} \\ \cdots \\ \mathbf{f}_{n\cdot 4^{i_0+1}}^{i_0+2} \end{pmatrix} \end{aligned}$$

and analogous expression for f^{i_0+2}, with $\mathbf{f}$ instead of $\widehat{\mathbf{f}}$. Therefore

$$\left\| f^{i_0+2} - \widehat{f}^{i_0+2} \right\| = \left\| \begin{pmatrix} u(f_{4p-3}^{i_0+1} - \widehat{f}_{4p-3}^{i_0+1}) \\ u(f_{4p-2}^{i_0+1} - \widehat{f}_{4p-2}^{i_0+1}) \\ u(f_{4p-1}^{i_0+1} - \widehat{f}_{4p-1}^{i_0+1}) \\ u(f_{4p}^{i_0+1} - \widehat{f}_{4p}^{i_0+1}) \end{pmatrix} \right\| = \frac{1}{2} \left\| \begin{pmatrix} \mathbf{f}_p^{i_0+1} - \widehat{\mathbf{f}}_p^{i_0+1} \\ \mathbf{f}_p^{i_0+1} - \widehat{\mathbf{f}}_p^{i_0+1} \\ \mathbf{f}_p^{i_0+1} - \widehat{\mathbf{f}}_p^{i_0+1} \\ \mathbf{f}_p^{i_0+1} - \widehat{\mathbf{f}}_p^{i_0+1} \end{pmatrix} \right\|,$$

whence

$$\begin{aligned} \left\| f^{i_0+2} - \widehat{f}^{i_0+2} \right\|_2 &= \left\| \mathbf{f}_p^{i_0+1} - \widehat{\mathbf{f}}_p^{i_0+1} \right\|_2 = \left\| f^{i_0+1} - \widehat{f}^{i_0+1} \right\|_2, \\ \left\| f^{i_0+2} - \widehat{f}^{i_0+2} \right\|_\infty &= \frac{1}{2} \left\| f^{i_0+1} - \widehat{f}^{i_0+1} \right\|_\infty \end{aligned}$$

Repeating this procedure we finally obtain

$$\begin{aligned} \|f^i - \widehat{f}^i\|_2 &= \left\| f^{i_0+1} - \widehat{f}^{i_0+1} \right\|_2 = |w_{k_0}^{i_0}| \cdot \|m_r\|_2, \\ \|f^i - \widehat{f}^i\|_\infty &= \frac{1}{2^{i-i_0-1}} \left\| f^{i_0+1} - \widehat{f}^{i_0+1} \right\|_\infty = |w_{k_0}^{i_0}| \cdot \|m_r\|_\infty. \end{aligned}$$

3 Optimal wavelets

We have two evaluate the norms $\|m_r\|_2$ and $\|m_r\|_\infty$, $r \in \{1, 2, 3\}$, for the two classes of wavelets, namely for

$$m = \begin{pmatrix} \alpha_1 & \alpha_2 & -0.5 - \alpha_1 - \alpha_2 \\ 0.5 & 0.5 & 0.5 \\ \alpha_2 & -0.5 - \alpha_1 - \alpha_2 & \alpha_1 \\ -0.5 - \alpha_1 - \alpha_2 & \alpha_1 & \alpha_2 \end{pmatrix},$$

with (α_1, α_2) on the ellipse E_1 and respectively for

$$m = \begin{pmatrix} \alpha_1 & \alpha_2 & 0.5 - \alpha_1 - \alpha_2 \\ -0.5 & -0.5 & -0.5 \\ \alpha_2 & 0.5 - \alpha_1 - \alpha_2 & \alpha_1 \\ 0.5 - \alpha_1 - \alpha_2 & \alpha_1 & \alpha_2 \end{pmatrix},$$

with (α_1, α_2) on the ellipse E_2. The calculations show that $\|m_r\|_2 = 1$, for $r \in \{1, 2, 3\}$, for both classes of wavelets. In conclusion, no matter which wavelet we choose, we obtain the same l^2-norm of the compression error. In the following we will focus on the norm $\|m_r\|_\infty$, which equals

$$\max \left\{ \frac{1}{2}, |\alpha_1|, |\alpha_2|, \left| \varepsilon \frac{1}{2} + \alpha_1 + \alpha_2 \right| \right\}, \text{ for all } r \in \{1, 2, 3\}, \tag{4}$$

where $\varepsilon = 1$ for $(\alpha_1, \alpha_2) \in E_1$ and $\varepsilon = -1$ for $(\alpha_1, \alpha_2) \in E_2$.
For the ellipse E_1 we have

$$\alpha_2^{1,2} = \frac{-1 - 2\alpha_1 \pm \sqrt{5 - 4\alpha_1 - 12\alpha_1^2}}{4},$$
$$\left|\frac{1}{2} + \alpha_1 + \alpha_2^1\right| = \left|\alpha_2^2\right|, \ \left|\frac{1}{2} + \alpha_1 + \alpha_2^2\right| = \left|\alpha_2^1\right|, \text{ so}$$

$$\|m_r\|_\infty = \max\left\{\frac{1}{2}, |\alpha_1|, |\alpha_2^1|, |\alpha_2^2|\right\}, \text{ with } \alpha_1 \in \left[-\frac{5}{6}, \frac{1}{2}\right].$$

A graph of the ellipse E_1 immediately allows us to state that $\|m_r\|_\infty$ attains its minimum value $\frac{1}{2}$ for

$$(\alpha_1, \alpha_2) \in \left\{\left(-\frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}\right)\right\}$$

and its maximum value $\frac{5}{6}$ for

$$(\alpha_1, \alpha_2) \in \left\{\left(-\frac{5}{6}, \frac{1}{6}\right), \left(\frac{1}{6}, \frac{1}{6}\right), \left(\frac{1}{6}, -\frac{5}{6}\right)\right\}.$$

Similarly, for E_2, $\|m_r\|_\infty$ attains its minimum value $\frac{1}{2}$ for

$$(\alpha_1, \alpha_2) \in \left\{\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}\right)\right\}$$

and its maximum value $\frac{5}{6}$ for

$$(\alpha_1, \alpha_2) \in \left\{\left(-\frac{1}{6}, \frac{5}{6}\right), \left(\frac{5}{6}, -\frac{1}{6}\right), \left(-\frac{1}{6}, -\frac{1}{6}\right)\right\}.$$

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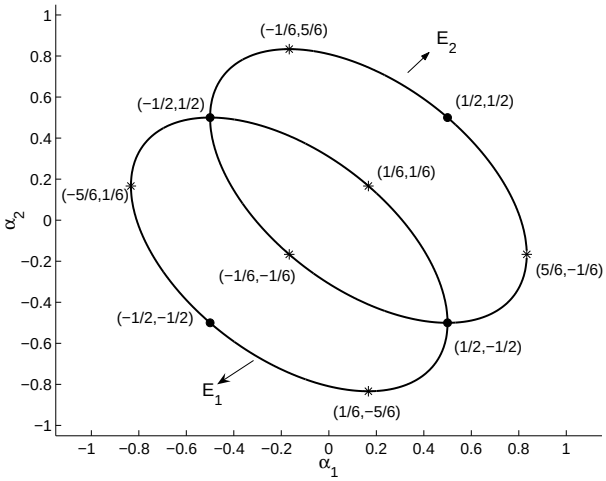

Figure 2: The ellipses E_1 and E_2 and the points (α_1, α_2) which lead to maximum $*$ and minimum $\bullet$ of the norm $\left\| f^i - \widehat{f^i} \right\|_\infty$.

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