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DIVIDE-AND-CONQUER PIECEWISE LINEAR APPROXIMATION OF GAIN AND PHASE EVALUATION

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Abstract: Certain approaches to phase approximation ask as a first step in implementation for a piecewise linear fitting of gain. When the breakpoints are given, there are few ways to determine the slopes of the broken line approximation. In this paper we determine the breakpoints using a divide-and-conquer approach. Results to piecewise linear approximation of gain and of the resulting phase approximation are presented. For this purpose, we consider certain frequency responses, then the gain data are corrupted or altered.

Keywords: *piecewise linear approximation, divide-and-conquer, phase approximation, Hilbert transform.*

I. INTRODUCTION

Hilbert transform and the related Bode relationships [1] have been recognized as very important methods in circuit theory, communications and control science. Their sampled derivations have been encountered in different applications from science and engineering. In some situations the domain is restricted or other explicit conditions are imposed. A critical issue is related to the singularities involved in the Hilbert transform computation, since we are confronted with an improper integral (Section II). If the integral cannot be evaluated in a closed form, as it is the case with discrete input data, numerical implementation is in general complicated [2], as localized errors in gain should lead to localized errors in phase approximation. Hilbert transform has the advantage of not requiring derivatives, but the serious disadvantage that it is not a bounded operator L^∞ to L^∞. To solve the problem, different approaches for gain-phase relationships in logarithmic frequency domain have been proposed. A suitable change of variable can give the bounded operator (6) from L^r to L^∞ for any $r > 1$ [3].

In many applications the goal is to obtain the phase at any desired frequency. To this end, there were proposed many approaches for phase approximation. Almost all of them are based on some quadrature formulas of Hilbert transform, where the knots of quadrature are specified a priori. However, in practical situations the values of gain are available only at certain frequencies, which may not be the exact knots of quadrature formula. The method proposed in [4] can deal with gain samples at arbitrary frequencies, but it requests for a piecewise linear approximation (PWLA) of gain. Although one can provide such PWLA by using all available gain samples, a PWLA with a reduced number of slopes and breakpoints is more attractive in implementation, as the number of added terms is much smaller.

The goal of this paper is to show how to determine the breakpoints of a piecewise linear fitting of gain by using a divide-and-conquer approach, then to use this technique for computing a phase approximation. For this purpose we first remind Hilbert transform and Bode relationships (Section II), then the gain non-compact phase approximation method is recalled (Section III). In Section IV we present the divide-and-conquer piecewise method, respectively in Section V the framework is illustrated. Finally numerical examples are provided (Section VI).

II. BODE RELATIONSHIPS AND HILBERT TRANSFORM

Let us consider $H(j\omega)$ the Fourier transform of a causal function $h(t)$:

$$H(j\omega) = \int_0^\infty h(t)e^{-j\omega t}dt = R(\omega) + jI(\omega), \quad (1)$$

then we have [1]

$$R(\omega) = R(\infty) - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{I(\omega)}{y-\omega}dy, \quad (2)$$

$$I(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R(y)}{y-\omega}dy, \quad (3)$$

which establish the Hilbert pair of $R(\omega)$ and $I(\omega)$. One can easily obtain the gain-phase relations (or Bode relationships) from (2) and (3) directly by taking logarithms [5], after fulfilling the requirements needed to satisfy the right half plane analyticity conditions of the Hilbert transform, i.e. the stable and minimum phase conditions. Under the assumption that $H(s)$ is not only analytic, but has no zeros for $Re(s) \geq 0$, then:

$$\ln H(j\omega) = \alpha(\omega) + j\beta(\omega), \quad (4)$$

will also be analytic in the right-hand plane. Thus the phase $\beta(\omega)$ will be uniquely determined from the gain (in nepers)

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$$\begin{aligned}\beta(\omega) &\overset{(3)}{\overbrace{=}} \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\alpha(y)}{y-\omega}dy \overset{(7)}{\overbrace{=}} \frac{1}{\pi}\int_{-\infty}^{\infty}\alpha'(y)\ln\left|\frac{y+\omega}{y-\omega}\right|dy \overset{(9)}{\overbrace{=}} \frac{1}{\pi}\int_{-\infty}^{\infty}\alpha''(y)\times\left(2\omega-\omega\ln\left|\omega^2-y^2\right|-y\ln\left|\frac{y+\omega}{y-\omega}\right|\right)dy\\ &\overset{(11)}{\overbrace{=}} \frac{1}{\pi}\int_{-\infty}^{\infty}\sum_n a_n\left[\delta(y)-\delta\left(y-\omega_n\right)\right]\times\left(2\omega-\omega\ln\left|\omega^2-y^2\right|-y\ln\left|\frac{y+\omega}{y-\omega}\right|\right)dy\\ &= \frac{1}{\pi}\sum_n a_n\left(2\omega-\omega\ln\left|\omega^2\right|-0-2\omega+\omega\ln\left|\omega^2-\omega_n^2\right|+\omega_n\ln\left|\frac{y+\omega}{y-\omega}\right|\right)\equiv\beta_a(\omega). \qquad (5)\end{aligned}$$

$\alpha(\omega)$:

$$\beta(\omega) = \frac{2\omega}{\pi}\int_0^{\infty}\frac{\alpha(y)-\alpha(\omega)}{y^2-\omega^2}dy. \qquad (6)$$

III. GAIN WITH NON-COMPACT SUPPORT

The formula between the imaginary and real parts of a complex function of real frequency as expressed in (3) can be rewritten in many ways [4]. By integrating the right member of (3) by parts we find:

$$I(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}R'(y)\ln\left|\frac{y+\omega}{y-\omega}\right|dy, \qquad (7)$$

provided

$$\lim_{y\to\infty}\frac{R(y)}{y} = 0. \qquad (8)$$

Alternatively, we can continue by integrating the right member of (8) by parts, i.e. a double integration by parts of the right member of (3) and the integrand will be:

$$R''(y)\left(2\omega-\omega\ln\left|\omega^2-y^2\right|-y\ln\left|\frac{y+\omega}{y-\omega}\right|\right), \qquad (9)$$

provided

$$\lim_{y\to\infty}\frac{R(y)}{y}=0, \quad \text{and} \quad \lim_{y\to\infty}R'(y)<\infty. \qquad (10)$$

Previous relationships are seldom integrated analytically and in practice it is customary to use approximations to find the relationship between phase and gain. An idea is to use straight-line segments so that the second derivative $\alpha''(\omega)$ is a set of impulses [4]. Gain functions will satisfy the following:

- Second derivative consists of groups of two impulses;
- Each group has a positive impulse at the origin and a negative impulse at a frequency denoted by ω_n;
- Only positive ω_n's need to be considered.

The second derivative of the gain is given by:

$$\alpha''(\omega) \approx \sum_n[\delta(\omega)-\delta(\omega-\omega_n)]. \qquad (11)$$

It follows successively the relationships presented in (5). Finally we get:

$$\beta(\omega) \approx \beta_a(\omega)=\frac{1}{\pi}\sum_n a_n\omega_n\phi\left(\frac{\omega}{\omega_n}\right) \qquad (12)$$

where

$$\phi(v) = (v+1)\ln|v+1|+(v-1)\ln|v-1|-2v\ln|v|.$$

Remarks:

1. The a_n numbers are determined by a broken-line approximation to the gain-versus-arithmetic-frequency characteristic;
2. This procedure cannot be employed when the gain characteristic has slopes different from zero, when frequency is near zero or at very high frequency.

To compute $\beta_a(\omega)$ one can follow the steps:

- Given the pairs frequency-gain samples $(\omega_i, \alpha(\omega_i))$, $i=\overline{1,I}$, find the piecewise-linear approximation $(\omega_n, \alpha(\omega_n))$, $n=\overline{1,N}$; for ω_0, $\alpha(\omega_0)$ is needed or should be evaluated;
- For $n=\overline{0,N-1}$, compute slopes s_n as follows: $s_n = [\alpha(\omega_{n+1})-\alpha(\omega_n)]/(\omega_{n+1}-\omega_n)$;
- $a_0=s_0$ and $a_n=s_{n-1}-s_n$ for $n=\overline{1,N-1}$;
- Compute $\beta_a(\omega)$ from (12), by summing of n from 0 to $N-1$.

IV. DIVIDE-AND-CONQUER PIECEWISE METHOD

There have been many piecewise linear approximations presented in literature. Usually any PWLA is described by its breakpoints and slopes, however the major issue is the selection of breakpoints. If the breakpoints are known, optimal slopes can be obtained [6, 7], and the PWLA results. However, when the breakpoints are not known, a certain random selection of breakpoints does not guarantee always that by applying optimal approaches one can obtain an overall optimal or even rather a good approximation. In the following the goal is to show a way how to determine the breakpoints of a piecewise linear fitting. The naive solution of this problem consists in computing of all successive slopes for all available points. Then one can detect significant changes between successive slopes, and when such change is detected, a new breakpoint is set. Such a strategy can be implemented, but it may require a great amount of computation, especially when the number of available points is very large.

To improve on the naive algorithm, we shall make use of a powerful technique, called divide-and-conquer algorithm. Such procedure if often encountered in science and technology, e.g. FFT (Fast Fourier Transform) or quicksort algorithms [8, 9]. Basically, it consists in partitioning the problem into two parts, then solving the parts independently, i.e. conquered individually. Finally, the results are put back together in some way.

For our purpose, we shall now describe the basic module of the algorithm:

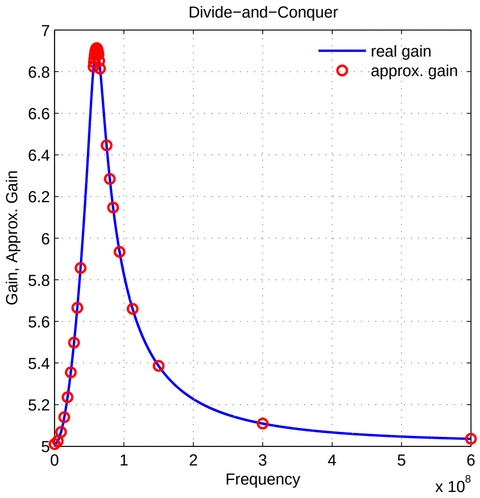

Figure 1. Gain (-) and gain samples (o) obtained by divide-and-conquer PWLA.

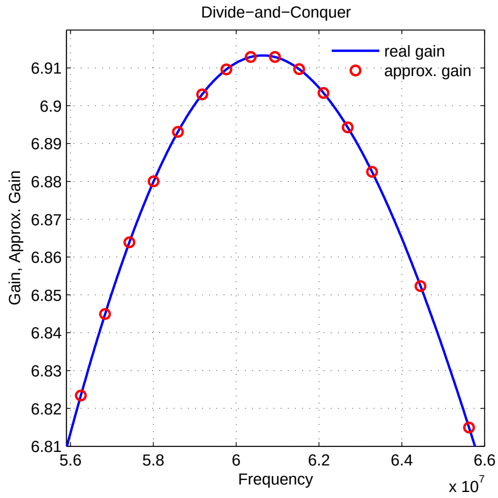

Figure 2. Gain (-) and gain samples (o) obtained by divide-and-conquer PWLA (zoom of Fig. 1).

1. Let us consider two breakpoints, usually denoted by the smallest point x_{min} and the largest point x_{max}, and we compute the slope for these two points:

$$\frac{F(x_{max}) - F(x_{min})}{x_{max} - x_{min}};$$

2. then we select the point from the middle of the interval

$$x_{mean} = \frac{x_{max} + x_{min}}{2};$$

3. and compute the slopes of function for the middle point

$$\frac{F(x_{max}) - F(x_{mean})}{x_{max} - x_{mean}};$$

$$\frac{F(x_{mean}) - F(x_{min})}{x_{mean} - x_{min}}.$$

If these new slopes are very close to the previous slope, then there is no need to add the middle point; otherwise the middle point is added to the set of breakpoints. This completes the basic module of the PWLA algorithm.

For the given set of data, we shall start with the minimum and the maximum available point, and they both define the first set of breakpoints. We apply the basic module of the algorithm and we get the second set of breakpoints. Recursively we apply the basic module for the obtained set of breakpoints, to get a new set of breakpoints. This procedure is applied until a satisfactory PWLA is reached.

Example: The gain together with its divide-and-conquer PWLA are shown in Fig. 1, for the frequency $\omega \in [0,\ 6 \cdot 10^8]$ Hz. We obtained 32 points by divide-and-conquer method that provide us a good PWLA of gain. A zoom of the PWLA for the highest values of gain is also depicted in Fig. 2.

We shall note that this approach is always convergent since in practical situations we have a finite number of available points. However, it may not converge always to a satisfactory approximation. This may happen when the function to be approximated has many local minimum or maximum points (gain with multiple peaks). In such situation, we suggest to compute the mean squared error between the function and its approximation. Thus one can decide whether the recursive process has been finished properly.

V. FRAMEWORK

Before proceedings with experimental results, we have to consider some preliminary discussions. First we recalled that the technique mentioned in Section III cannot be used when the gain characteristic has slopes different from zero near zero or at very high frequency. According to Newton, Gould and Kaiser [4], this restriction is not a great problem: the phase characteristic of a factor in a transfer function whose gain characteristic does not have the above mentioned property can be evaluated alone. This may be a sensitive problem only when a small number of gain samples are available.

Now we are going to test the given approach. For phase approximation in linear frequency domain using non-compact gain technique, Bode circuit function [1] was modified as bellow:

$$H(s) = Z_{in}(s) = \frac{Es+F|}{|As} + \frac{1|}{|Bs} + \frac{1|}{|Cs} + \frac{1|}{|D} \quad (13)$$

by inserting a new zero in the transfer function in order to smooth the gain behavior at zero frequencies. Note that E and F must be positive in order to keep the transfer function as a minimum-phase one. In what follows we shall demonstrate the required slopes conditions for these modified transfer functions. Equation (13) can be rewritten as:

$$H(s) = Z_{in}(s)$$
$$= \frac{BCDEs^3 + B(CDF+E)s^2 + (BF+DE)s + DF}{ABCDs^3 + ABs^2 + D(A+C)s + 1}.$$

The corresponding magnitude for the frequency response

$$Z_{in}(s) = \frac{(R_0+R_1)s^3 + \frac{R_0C_1L_2+R_0R_1C_1C_3R_4+R_1C_1L_2+L_2C_3R_4}{C_1L_2C_3R_4}s^2}{s^3+\frac{L_2+R_1C_3R_4}{L_2C_3R_4}s^2+\frac{C_1R_4+R_1C_1+C_3R_4}{C_1L_2C_3R_4}s+\frac{1}{C_1L_2C_3R_4}} \quad (14)$$
$$+\frac{\frac{R_0C_1R_4+R_0R_1C_1+R_0C_3R_4+R_1C_1R_4+L_2}{C_1L_2C_3R_4}s+\frac{R_0+R_4}{C_1L_2C_3R_4}}{\;}.$$

function $H(j\omega) = H(s)|_{s=j\omega}$ is given by:

$$|H(j\omega)| = \frac{\sqrt{B^2C^2D^2E^2\omega^6 + B(BC^2D^2F^2+BE^2}}{\sqrt{A^2B^2C^2D^2\omega^6 + AB(AB-2ACD^2}}$$
$$\frac{\overline{-2CD^2E^2)\omega^4 + (B^2F^2+D^2E^2-2BCD^2F^2)\omega^2 + D^2F^2}}{\overline{-2C^2D^2)\omega^4 + (A^2D^2+2ACD^2+C^2D^2-2AB)\omega^2+1}}.$$

The first order logarithmic gain derivative is given in [10]. The gain characteristic must have zero slope at zero and at very high frequencies. We shall evaluate these slopes and impose the needed conditions to demonstrate that the modified Bode's transfer functions can be used as test functions for non-compact gain technique. The slope at the infinity should satisfy:

$$\lim_{\omega\to\infty} \alpha'(\omega) = 0,$$

so the $12th$ order term from $U(\omega)V(\omega)$ must has nonzero value:

$$A^2B^4C^4D^4E^2 \neq 0,$$

from where we have: $A \neq 0, B \neq 0, C \neq 0, D \neq 0, E \neq 0$.

The slope at zero should satisfy:

$$\lim_{\omega\to 0} \alpha'(\omega) = 0,$$

so:

- the free term (the coefficient for ω^0) from $U(\omega)V(\omega)$ must have nonzero value:
$$D^2F^2 \neq 0,$$
from where we have $D \neq 0, F \neq 0$.
- the free term from $U'(\omega)V(\omega) - V'(\omega)U(\omega)$ must be zero; because $U'(\omega)$ and $V'(\omega)$ are odd functions, $U(\omega)$ and $V(\omega)$ are even functions results that $U'(\omega)V(\omega) - V'(\omega)U(\omega)$ is an odd one, so:
$$U'(\omega)V(\omega) - V'(\omega)U(\omega)\big|_{\omega=0} = 0.$$

It follows that the necessary and sufficient condition for the modified Bode's transfer functions to be applied in the case of gain with non-compact support is: $A \neq 0, B \neq 0, C \neq 0, D \neq 0, E \neq 0, F \neq 0$, or equivalent:

$$A\cdot B\cdot C\cdot D\cdot E\cdot F \neq 0. \quad (15)$$

The modified Bode transfer function (13) corresponds to the circuit from Fig. 3. Its input impedance is:

$$Z_{in}(s) = R_0 + \frac{1|}{\left|\frac{sC_1}{1+sR_1C_1}\right.} + \frac{1|}{|sL_2} + \frac{1|}{|sC_3} + \frac{1|}{|R_4} \quad (16)$$

The extended form of the input impedance is presented in (14).

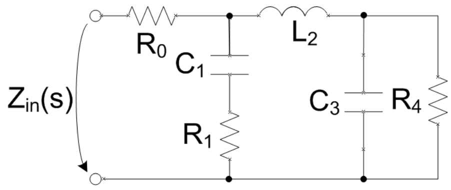

Figure 3. Modified Bode circuit.

Threshold value	No. of points	Threshold value	No. of points
$2.170e-009$	32	$6.247e-010$	118
$1.860e-009$	34	$5.941e-010$	125
$1.674e-009$	41	$5.632e-010$	131
$1.507e-009$	44	$5.320e-010$	137
$1.424e-009$	46	$5.005e-010$	139
$1.382e-009$	53	$4.687e-010$	144
$1.315e-009$	56	$4.366e-010$	148
$1.269e-009$	60	$4.042e-010$	151
$1.222e-009$	65	$3.715e-010$	169
$1.149e-009$	68	$3.385e-010$	205
$1.022e-009$	71	$3.052e-010$	244
$9.415e-010$	73	$2.716e-010$	262
$8.587e-010$	75	$2.377e-010$	279
$8.305e-010$	77	$2.035e-010$	302
$8.020e-010$	80	$1.690e-010$	432
$7.732e-010$	83	$1.342e-010$	557
$7.441e-010$	87	$9.910e-011$	646
$7.147e-010$	90	$6.370e-011$	1145
$6.850e-010$	104	$2.800e-011$	2437
$6.550e-010$	113	$1.100e-011$	6207

Table 1. The relationship between the threshold value and the number of breakpoints – $\omega \in [0,\ 6\cdot 10^8]$.

VI. NUMERICAL EXAMPLES

The considered values for the components are as follows: $R_0 = 100\Omega$, $R_1 = 50\Omega$, $C_1 = 56$pF, $L_2 = 5\mu$H, $C_3 = 56$pF and respectively $R_4 = 50\Omega$.

VI.1 Threshold value vs number of breakpoints

We are interested to observe how variate the breakpoints number for the piecewise linear approximation (PWLA), using the divide-and-conquer approach, regarding the threshold value.

For $\omega \in [0,\ 6\cdot 10^8]$, in Table 1 we show the relation between the threshold value and the number of the PWLA breakpoints, and in Fig. 4 this relationship is plotted.

In Table 2, respectively in Fig. 5 the relation between the threshold value and the number of the PWLA breakpoints is illustrated for $\omega \in [0,\ 10^9]$.

From Figs. 4 and 5 respectively Tables 1 and 2, we can conclude that the number of breakpoints for the PWLA increased exponentially by decreasing the threshold value. However, for a given threshold value there is almost no difference between the obtained number of breakpoints and the frequency range used for approximation. Instead,

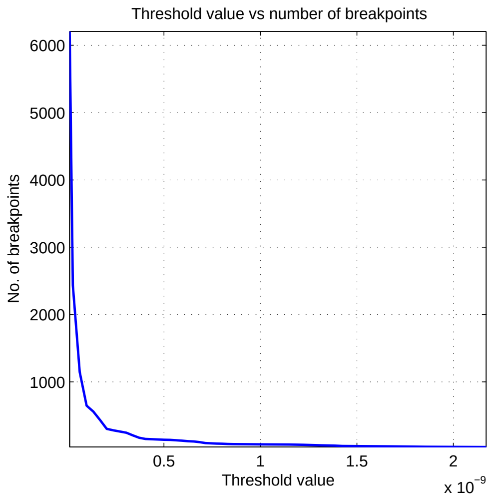

Figure 4. The relationship between the threshold value and the number of breakpoints – $\omega \in [0,\ 6 \cdot 10^8]$.

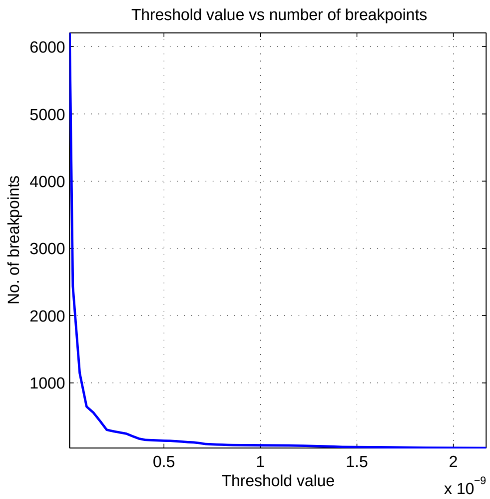

Figure 5. The relationship between the threshold value and the number of breakpoints – $\omega \in [0,\ 10^9]$.

Threshold value	No. of points	Threshold value	No. of points
$4.890e-010$	160	$2.334e-010$	326
$4.799e-010$	162	$2.189e-010$	339
$4.700e-010$	164	$2.040e-010$	353
$4.637e-010$	166	$1.887e-010$	366
$4.484e-010$	168	$1.730e-010$	382
$4.394e-010$	170	$1.569e-010$	444
$4.295e-010$	173	$1.404e-010$	576
$4.187e-010$	176	$1.235e-010$	659
$4.070e-010$	179	$1.062e-010$	688
$3.714e-010$	185	$9.740e-011$	717
$3.615e-010$	188	$8.850e-011$	750
$3.512e-010$	190	$7.950e-011$	822
$3.405e-010$	194	$7.040e-011$	1032
$3.237e-010$	203	$6.120e-011$	1258
$3.120e-010$	212	$5.190e-011$	1388
$2.999e-010$	222	$4.250e-011$	1520
$2.874e-010$	250	$3.300e-011$	2285
$2.745e-010$	269	$2.340e-011$	2914
$2.612e-010$	291	$1.370e-011$	5396
$2.475e-010$	317	$3.900e-012$	19870

Table 2. The relationship between the threshold value and the number of breakpoints – $\omega \in [0,\ 10^9]$.

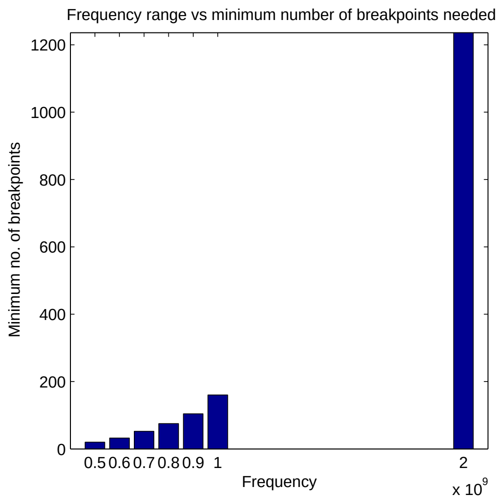

Figure 6. The relationship between the frequency range and the minimum number of breakpoints needed.

the frequency range influence the maximum value for the threshold, and implicitly the minimum number of PWLA breakpoints. The number of the breakpoints increase exponentially by increasing the frequency range, as one can see in Fig. 6.

VI.2 Approximated phase vs number of breakpoints

In what follows, we are interested to see how the non-compact gain technique [4], for phase approximation from the gain samples in the linear frequency domain, behaves regarding the number of the PWLA breakpoints. For an exact evaluation for the difference between phase and the corresponding approximated one, we have been evaluated the squared error.

For $\omega \in [0,\ 6 \cdot 10^8]$, the phase approximations for different values of the breakpoints are illustrated in Fig. 7, together with the real phase, and the corresponding square error is shown in Fig. 8.

For $\omega \in [0,\ 10^9]$, the phase approximations for different values of the breakpoints are illustrated in Fig. 9, together with the real phase, and the corresponding square error is shown in Fig. 10.

From Figs. 7 and 9 we can conclude that the number of breakpoints used for PWLA influence the quality of the phase approximation. For a given frequency range, increasing the number of breakpoints, the square error between phase and approximated phase decreases. However, for the same number of breakpoints, as we increase the frequency range, the phase approximation behaves better. This can be explain by the fact that the gain slopes tend 'much better' to zero, at zero and at very high frequencies, thus the quality of the approximation increases. If the number of the breakpoints is too high, then the evaluation time is also high, which may be not desirable. Thus the investigator must impose the highest admissible value for the square error, bellow which there is no need to increase the number of the breakpoints.

VI.3 Gain corrupted data

The above illustrations were done for frequency response data generated by ideal responses of the transfer function. The ideal responses could also be corrupted by various

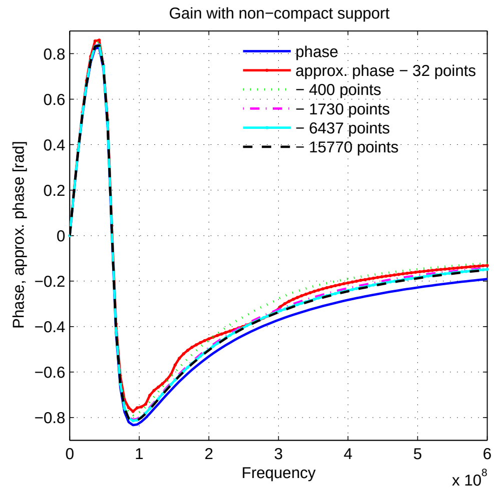

Figure 7. Phase (-) and approximated phase – $\omega \in [0,\ 6 \cdot 10^8]$.

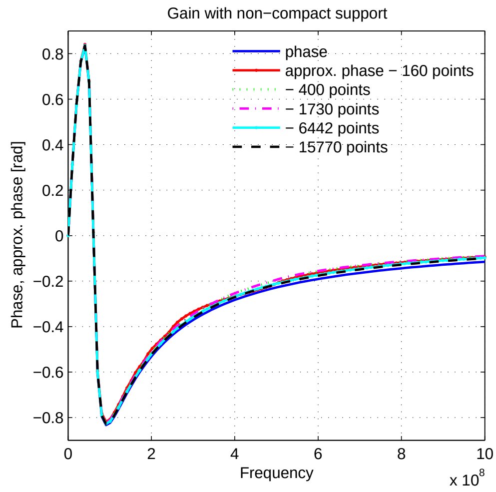

Figure 9. Phase (-) and approximated phase – $\omega \in [0,\ 10^9]$.

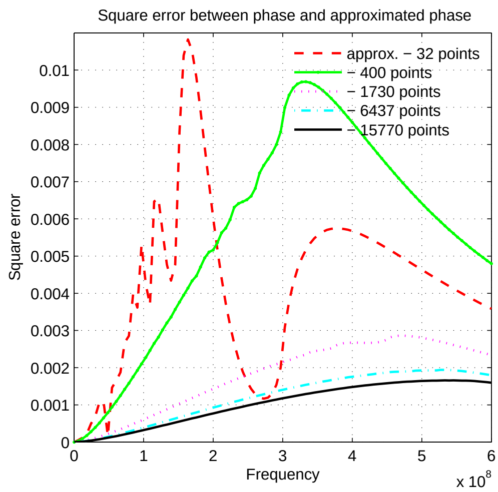

Figure 8. Square error – $\omega \in [0,\ 6 \cdot 10^8]$.

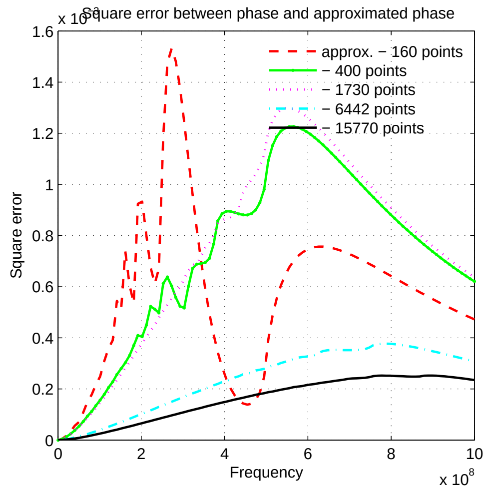

Figure 10. Square error – $\omega \in [0,\ 10^9]$.

types and levels of noise. In what follows we will evaluate the non-compact support gain technique performances, for phase approximation in the linear frequency domain, when the circuit function is noise corrupted.

We will start with the case for which on the true parameters we add a percent error (tolerance). The true parameters and their associated nominal values will be available for use as initial estimates [11]. A percentage error of

1. $\pm 1\%$;
2. $\pm 5\%$;
3. $\pm 10\%$.

will be added to test the the phase approximation method in the linear frequency domain.

To compare the effect of the number of breakpoints used for PWLA regarding the approximated phase, when the nominal values of the circuit function (14) are tolerated, we will use the well known norms L^1, L^2 and L^∞. All the tables contain on the left the value of the tolerance, then the number of breakpoints used for PWLA approximation, and on the last three right columns the values obtained for the L^1, L^2, respectively L^∞ norms.

For $\omega \in [0,\ 6 \cdot 10^8]$, the results are shown in Table 3. To compare the results obtained for the approximated phase when the nominal values of the parameters are altered by the corresponding tolerance, we also show the results for 0% tolerance. In Table 4 are shown the results for $\omega \in [0,\ 10^9]$.

From Tables 3 and 4 we can conclude that the behavior of phase approximation using gain altered data by a percent error is the same as in the case of non-corrupted gain data: increasing the number of breakpoints, the error between phase and approximated one decreases, and also, for the same number of breakpoints, considering the same tolerance, increasing the frequency range, the error decreases.

Another case, is the one when the circuit functions are corrupted by noise. At any ideal frequency response, $H(j\omega_k)$, noise corrupted data were taken as:

$$\overline{H}(j\omega_k) = H(j\omega_k) + \Delta H(j\omega_k), \qquad (17)$$

where $\mathrm{Re}\{\Delta H(j\omega_k)\}$ and $\mathrm{Im}\{\Delta H(j\omega_k)\}$, are generated using a normal distribution with zero mean and σ^2 vari-

Tol.	Points	L^1	L^2	L^∞
	32	6.182e−002	6.507e−002	1.040e−001
	400	7.078e−002	7.472e−002	9.842e−002
0%	1730	4.004e−002	4.247e−002	5.348e−002
	6437	3.283e−002	3.493e−002	4.407e−002
	15770	3.017e−002	3.215e−002	4.071e−002
	31	6.417e−002	6.850e−002	1.111e−001
	400	7.289e−002	7.775e−002	1.016e−001
1%	1730	4.403e−002	4.671e−002	5.643e−002
	6437	3.732e−002	3.954e−002	4.689e−002
	15770	3.485e−002	3.691e−002	4.365e−002
	36	6.041e−002	6.337e−002	9.792e−002
	400	6.922e−002	7.261e−002	9.523e−002
−1%	1730	3.746e−002	3.983e−002	6.822e−002
	6437	3.000e−002	3.228e−002	6.461e−002
	15770	2.725e−002	2.952e−002	6.329e−002
	40	8.207e−002	8.799e−002	1.834e−001
	400	9.063e−002	9.602e−002	1.914e−001
5%	1730	6.614e−002	7.149e−002	2.047e−001
	6437	6.046e−002	6.620e−002	2.077e−001
	15770	5.836e−002	6.430e−002	2.088e−001
	9	9.345e−002	1.314e−001	5.891e−001
	400	6.346e−002	7.585e−002	3.003e−001
−5%	1730	3.113e−002	5.044e−002	2.819e−001
	6437	2.475e−002	4.591e−002	2.762e−001
	15770	2.421e−002	4.556e−002	2.755e−001
	41	1.058e−001	1.183e−001	3.812e−001
	400	1.132e−001	1.245e−001	3.867e−001
10%	1730	9.264e−002	1.062e−001	3.975e−001
	6437	8.785e−002	1.023e−001	3.998e−001
	15770	8.609e−002	1.009e−001	4.007e−001
	24	5.288e−002	1.018e−001	5.699e−001
	400	6.288e−002	1.050e−001	5.605e−001
−10%	1730	4.423e−002	9.317e−002	5.387e−001
	6437	4.550e−002	9.354e−002	5.319e−001
	15770	4.608e−002	9.373e−002	5.311e−001

Table 3. $\omega \in [0,\ 6\cdot 10^8]$ *– tolerance altered parameters.*

Tol.	Points	L^1	L^2	L^∞
	160	2.268e−002	2.389e−002	3.913e−002
	400	2.557e−002	2.688e−002	3.502e−002
0%	1730	2.588e−002	2.732e−002	3.603e−002
	6442	1.454e−002	1.542e−002	1.942e−002
	15770	1.191e−002	1.267e−002	1.589e−002
	163	2.659e−002	2.804e−002	4.530e−002
	400	2.951e−002	3.067e−002	4.018e−002
1%	1730	2.984e−002	3.106e−002	4.037e−002
	6442	1.917e−002	1.989e−002	4.420e−002
	15770	1.669e−002	1.739e−002	4.505e−002
	154	1.992e−002	2.151e−002	6.098e−002
	400	2.289e−002	2.471e−002	5.902e−002
−1%	1730	2.322e−002	2.519e−002	5.880e−002
	6442	1.173e−002	1.391e−002	5.470e−002
	15770	9.349e−003	1.156e−002	5.379e−002
	173	4.452e−002	5.069e−002	2.106e−001
	400	4.668e−002	5.193e−002	2.120e−001
5%	1730	4.743e−002	5.247e−002	2.124e−001
	6442	3.818e−002	4.466e−002	2.158e−001
	15770	3.603e−002	4.303e−002	2.165e−001
	79	1.765e−002	3.513e−002	2.395e−001
	400	2.158e−002	3.705e−002	2.713e−001
−5%	1730	2.222e−002	3.736e−002	2.710e−001
	6442	1.786e−002	3.603e−002	2.663e−001
	15770	1.809e−002	3.644e−002	2.652e−001
	180	6.533e−002	7.983e−002	3.699e−001
	400	6.582e−002	8.023e−002	3.704e−001
10%	1730	6.779e−002	8.093e−002	3.714e−001
	6442	5.997e−002	7.518e−002	3.743e−001
	15770	5.998e−002	7.518e−002	3.743e−001
	90	3.333e−002	7.386e−002	5.323e−001
	400	3.425e−002	7.402e−002	5.259e−001
−10%	1730	3.456e−002	7.415e−002	5.257e−001
	6442	4.223e−002	7.778e−002	5.200e−001
	15770	4.588e−002	7.905e−002	5.187e−001

Table 4. $\omega \in [0,\ 10^9]$ *– tolerance altered parameters.*

ance ($N(0, \sigma^2)$), with

$$3\sigma = \frac{\eta}{100}\left|H\left(j\omega_k\right)\right|,$$

η being the level of noise [12]. Frequency response data will be generated using the modified Bode transfer function (14), with:

1. $\eta = 1\%$;
2. $\eta = 5\%$.

To compare the effect of breakpoints number used for PWLA regarding the quality of the approximated phase, when frequency response data are noise corrupted, we will use the same three norms L^1, L^2 and L^∞. We used the same threshold values as in the case for 0% tolerance.

For $\omega \in [0,\ 6\cdot 10^8]$, the results are shown in Table 5, for $\omega \in [0,\ 8\cdot 10^8]$ in Table 6, for $\omega \in [0,\ 10^9]$ in Table 7, respectively for $\omega \in [0,\ 2\cdot 10^9]$ the results are illustrated in Table 8.

Noise level	Points	L^1	L^2	L^∞
	203	5.087e−002	5.464e−002	8.726e−002
	3068	7.252e−002	7.652e−002	9.961e−002
$\eta = 1\%$	8402	1.350e+002	1.419e+002	1.725e+002
	15845	1.973e+001	2.074e+001	2.522e+001
	16251	7.846e+001	8.246e+001	1.003e+002
	250	3.839e−002	4.273e−002	6.589e−002
	3304	7.471e−002	7.882e−002	1.016e−001
$\eta = 5\%$	8000	8.193e−002	8.594e−002	1.166e−001
	15980	4.402e+001	4.627e+001	5.627e+001
	16255	1.740e+002	1.829e+002	2.224e+002

Table 5. $\omega \in [0,\ 6\cdot 10^8]$ *– noise corrupted data.*

Noise level	Points	L^1	L^2	L^∞
	1100	3.529e−002	3.707e−002	6.017e−002
	2337	4.566e−002	4.797e−002	6.234e−002
$\eta = 1\%$	7414	3.466e−002	3.679e−002	4.953e−002
	14272	6.995e+001	7.353e+001	8.940e+001
	16191	2.773e+001	2.914e+001	3.544e+001
	1147	3.508e−002	3.684e−002	5.943e−002
	27823	5.400e−002	5.704e−002	7.829e−002
$\eta = 5\%$	7470	2.811e−002	3.021e−002	4.696e−002
	15345	1.573e+002	1.653e+002	2.010e+002
	16378	7.442e+001	7.822e+001	9.513e+001

Table 6. $\omega \in [0,\ 8\cdot 10^8]$ *– noise corrupted data.*

Noise level	Points	L^1	L^2	L^∞
	794	1.815e−002	1.942e−002	3.108e−002
	2320	2.146e−002	2.275e−002	3.069e−002
$\eta = 1\%$	5287	2.555e−002	2.689e−002	3.493e−002
	12262	5.011e+001	5.267e+001	6.404e+001
	15993	5.762e+001	6.057e+001	7.365e+001
	1065	1.287e−002	1.451e−002	2.081e−002
	3707	1.614e−002	1.791e−002	2.518e−002
$\eta = 5\%$	6538	2.479e−002	2.611e−002	4.055e−002
	12247	1.120e+002	1.177e+002	1.431e+002
	16378	1.285e+002	1.351e+002	1.642e+002

Table 7. $\omega \in [0,\ 10^9]$ *– noise corrupted data.*

Noise level	Points	L^1	L^2	L^∞
$\eta = 1\%$	1637	$3.717e-003$	$5.383e-003$	$1.364e-002$
	3532	$5.228e-003$	$5.716e-003$	$1.152e-002$
	6363	$3.102e-003$	$3.614e-003$	$1.229e-002$
	13900	$2.479e+001$	$2.605e+001$	$3.169e+001$
	16219	$9.659e+001$	$1.015e+002$	$1.235e+002$
$\eta = 5\%$	1847	$1.433e-002$	$1.693e-002$	$3.135e-002$
	3780	$5.840e-003$	$6.735e-003$	$2.655e-002$
	7963	$5.627e-003$	$8.795e-003$	$3.044e-002$
	12096	$5.537e+001$	$5.820e+001$	$7.078e+001$
	16385	$2.160e+002$	$2.270e+002$	$2.761e+002$

Table 8. $\omega \in [0,\ 2\cdot 10^9]$ *– noise corrupted data.*

VII. CONCLUSION

In this paper we have presented a phase approximation using a piecewise linear fitting of gain. The slopes and the breakpoints of the broken line approximation are determined using a divide-and-conquer approach. Results in phase approximation are relatively accurate, even if frequency response data are corrupted by perturbations, and complexity of implementation is not very large. Comparisons with other numerical methods can be found in [10]. We have to note that this approach is always convergent since in practical situations we have a finite number of available points. However, it may not converge always properly, for instance when the gain has multiple peaks.

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