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A NEW ALGORITHM FOR FRACTIONAL BROWNIAN MOTION PROCESSES GENERATION

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Abstract: **The fractional Brownian motion random processes occur widely in nature and are a source of considerable interest in many fields. Sometimes the acquisition of data of this kind is laborious and time consuming. This is the reason why the problem of the generation of the fractional Brownian motion processes is of interest. We present in this paper an algorithm for the generation of fractional Brownian processes based on wavelets' theory and we analyze the proposed algorithm's performance. We compare the performance of the algorithm with the performance of state of the art algorithms for the generation of fractional Brownian motion random processes.**

Keywords: *fractional Brownian motion random process, wavelets, stationarity, communication traffic.*

I. INTRODUCTION

Flicker noise or 1/f noise (sometimes also called pink noise) is a random process with an average power spectral density (PSD) inversely proportional to the frequency of the random process. The term pink noise is sometimes used a little more loosely to refer to any noise with an average PSD inverse proportional with $|f|^{\alpha}$ with $0 < \alpha \leq 2$. These random signals are called pink-like noises. A pink-like noise with α=2 can be obtained by integrating a zero mean white noise with unitary variance. If the input process is a Gaussian white noise $G(t)$ with zero mean and variance σ^2, then the output process, obtained by integration, becomes a Brownian motion process, $B(t)$. By generalizing the notion of integration for fractional order, we can generate pink-like noises and fractional Brownian motion random processes [2, 3]. By derivation of the Gaussian pink-like noise, we obtain a fractionally Gaussian noise (fGn). Mandelbrot and Van Ness [4] proposed the name fractional noise (sometimes since called fractal noise) to emphasize that the exponent of the spectrum could take non-integer values and be closely related to fractional Brownian motion (fBm). Among other properties, the fBm possess the statistical self-similarity property [3, 5]. These fBm processes occur widely in nature and are a source of considerable interest in many fields: economics, study of fluctuations in solids, hydrology and communications. Many fluctuations in solids are fBm processes because their sample spectral density takes the form $|f|^{-1-2H}$, where H is the Hurst exponent and takes values in the interval $1/2 < H < 1$. This is, in fact, the condition of Long Range Dependency (LRD). Hurst found, researching in hydrology, the range of cumulated water flows to vary proportionately to t^H with $1/2 < H < 1$, where *t* denotes time. There is now ample evidence that fBm processes are present in a wide range of generalized data types appearing in communications, including many of those likely to form major components of telecommunications traffic in high-speed networks [6, 7, 8]. The aim of this paper is to propose a new algorithm for the generation of fBm random processes based on wavelets theory. This algorithm represents a generalization of the pink noise generation algorithm proposed in [17].

We discuss the mechanisms of fBm processes spectral analysis in section II. We show the connection between fBm processes and wavelets and we highlight the importance of the Hurst exponent in same section. We describe a wavelet based Hurst exponent estimation method and we justify our proposed generation method at the end of section II. Some simulation results and comparisons with other pink like noises generation methods are shown in section III. The last section contains the conclusions and some propositions for further research.

II. WAVELETS AND FRACTIONAL BROWNIAN PROCESSES

There is a strong connection between the fBm processes and the wavelets theory. The Hurst exponent can be estimated using wavelets. One of the best time-scale representations is the Continuous Wavelet Transform (CWT) [11]. The CWT of a random process $x(t)$ is expressed as [11]:

$$CWT_x(t,a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} x(s)\psi\left(\frac{s-t}{a}\right) ds, \qquad (1)$$

where the scale is denoted by a (a>0) and the function ψ (called mother wavelets-MW) has zero mean. The function $\psi_{t,a}(s) = \psi((s-t)/a)/\sqrt{a}$ is named analyzing wavelet. It is obvious that the expectation of the CWT of any random process equals zero. We compute the correlation of CWT:

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$$r_{T_x}(t,s;a)=E\{CWT_x(t,a)CWT_x(s,a)\}= \frac{1}{a}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E\{x(u)x(v)\}\psi\left(\frac{u-t}{a}\right)\psi\left(\frac{v-s}{a}\right)dudv. \quad (2)$$

If x is wide sense stationary then $r_x(u-v)=E\{x(u)x(v)\}$ and the correlation of its CWT becomes:

$$r_{T_x}(t,s;a)=a\int_{-\infty}^{\infty}r_x(a\tau)\,r_\psi\left(\tau-\frac{t-s}{a}\right)d\tau = =a\left(r_x(a\tau)*r_\psi(\tau)\right)\Big|_{\tau=\frac{t-s}{a}} = =\left(r_x(u)*r_\psi\left(\frac{u}{a}\right)\right)\Big|_{u=t-s} \quad (3)$$

fBm is a natural extension of ordinary Brownian motion, $B(t)$ [4]. It is a Gaussian zero-mean non stationary stochastic process $B_H(t)$, indexed by a single scalar parameter $0<H<1$, the usual Brownian motion being recovered from the specification $H=1/2$ ($\alpha=2H+1=2$) [5]. The non stationary character of fBm is observable from its auto-correlation structure [5]:

$$E\{B_H(t)B_H(s)\}=\frac{\sigma^2}{2}\left(|t|^{2H}+|s|^{2H}-|t-s|^{2H}\right), \quad (4)$$

where E stands for the statistical expectation operator and σ is a parameter for the selection of the processes' power. As non stationary process, fBm does not admit a power spectral density in the usual sense. However, it is possible to attach to it an average spectrum [9]:

$$S_{B_H}(f)=\sigma^2/|f|^{2H+1}. \quad (5)$$

fBm does have stationary increments [5], which means that the probability properties of the process $B_H(t+s)-B_H(t)$ only depends on the lag variable s. Moreover, this increment process is self-similar in the sense that, for any $a>0$ (with the convention $B_H(0)=0$):

$$B_H(at)\overset{distribution}{=}a^H B_H(t). \quad (6)$$

This self-similarity has as consequence that each individual realization of a fBm process is a fractal curve with the fractal dimension $D=2-H$ [5]. The increments of the fBm, which represent fGn, are second order stationary random processes with PSD of the form $1/|f|^{\alpha-2}$.

The expectation of the CWT of a fBm process equals zero. The correlation of the CWT of fBm process, which is no-stationary, can be computed by substituting in (2) the expression of the correlation in eq. (4):

$$r_{T_{B_H}}(t,s;a)=\frac{\sigma^2}{2a}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(|u|^{2H}+|v|^{2H}-|u-v|^{2H}\right) \psi\left(\frac{u-t}{a}\right)\psi\left(\frac{v-s}{a}\right)dudv=...= =-\frac{\sigma^2}{2a}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(|u-v|^{2H}\right)\psi\left(\frac{u-t}{a}\right)\psi\left(\frac{v-s}{a}\right)dudv \quad (7)$$

By comparing with eq. (2), it can be remarked that this is the correlation of the CWT of a stationary random process, $y(t)$, for which $E\{y(u)y(v)\}=|u-v|^{2H}$ depends only on the difference of the two moments of time u and v. So, it can be expressed in the form suggested by eq. (6) (where x is substituted by y):

$$r_{T_{B_H}}(t,s;a)=a\int_{-\infty}^{\infty}r_y(a\tau)\cdot r_\psi\left(\tau-\frac{t-s}{a}\right)d\tau = =-\frac{\sigma^2}{2}a^{2H+1}\int_{-\infty}^{\infty}|\tau|^{2H}\cdot r_\psi\left(\tau-\frac{t-s}{a}\right)d\tau. \quad (8)$$

In consequence, the CWT transforms some non stationary random processes (as the fBm processes are) into stationary processes whose PSD can be computed. By taking the Fourier transform of the auto-correlation in (8), Patrick Flandrin obtained in [10], the PSD of the fBm process considered, given in eq. (5). The CWT implementation requires high computational resources. This drawback can be reduced by bivariate sampling ($a=2^j, t=2^j k$), obtaining the Discrete Wavelet Transform (DWT) [12-14] which, unlike the CWT, is discrete in time and scale. The signal $x(t)$ can be reconstructed from DWT of $x(n)$ (obtained by sampling) by decomposition in wavelets series:

$$x(t)=\sum_{j=1}^{J}\sum_k d_{k,j}\psi_{k,j}(t)+\sum_k a_{k,J}\varphi_J(t) \quad (9)$$

where J indicates the number of DWT decomposition levels, $d_{k,j}$ represent detail coefficients (computed as scalar products between $x(t)$ and the wavelets $\psi_{k,j}(s)=\psi_{t,a}(s)\big|_{t=2^j k, a=2^j}=2^{-\frac{j}{2}}\psi(2^{-j}s-k)$) and $a_{k,J}$ represent approximation coefficients. DWT realizes an octave-based frequency analysis, the length of the bandwidth of the subband of details d_2 is half the length of the bandwidth of the subband of details d_1 and so on. So, the DWT is appropriate for the frequency analysis of the fBm random processes. Taking into account the fact that the DWT is obtained by sampling the CWT ($a=2^j, t=2^j k$), we can compute the expectation of the detail coefficients of a stationary random process $x(t)$ with mean μ_x:

$$\mu_{d_{k,j}} = E\{DWT_x(k,j)\} = E\{CWT_x(t,a)\}\big|_{t=2^j k,\, a=2^j} = 0, \quad (10)$$

k - any integer, $j = 1,..., J$.

Concerning the expectation of approximation coefficients:

$$\mu_{a_{k,J}} = 2^{\frac{J}{2}} \mu_x \quad (11)$$

The intra-scale correlation of detail coefficients can be computed by the particularization of eq. (3). The CWT of the fBm process is a stationary process and its correlation is expressed in Eq. (8) with $r_y(\tau) = |\tau|^{2H}$. Particularizing the CWT to the DWT by dyadic sampling ($a = 2^j, t = 2^j k$), the Eq. (8) becomes the expression of the intra-scale correlation of considered fBm process DWT detail coefficients [15]:

$$\rho_{d_{B_H}}(k,l;j) = 2^j \left(r_y\left(2^j \tau\right) * r_\psi(\tau) \right)\big|_{\tau=k-l}, \quad (12)$$

or in the case of orthogonal wavelets:

$$_o\rho_{d_{B_H}}(k,l;j) = 2^j \left(r_y\left(2^j \tau\right) \right)\big|_{\tau=k-l}. \quad (13)$$

Substituting the expression of the correlation of the process $y(t)$ into the last equation, we obtain the final expression of the intra-scale correlation of the fBm process DWT detail coefficients with orthogonal wavelets:

$$_o\rho_{d_{B_H}}(k;j) = 2^j \left(\left|2^j \tau\right|^{2H} \right)\Big|_{\tau=k} = 2^{j(2H+1)} |k|^{2H}, \quad (14)$$

which characterize a stationary random process. As the CWT, the “wavelet part” of DWT “makes stationary” the fBm random processes. The no stationarity part of the fBm process is conserved into the “scaling function part” of DWT. The variance of the detail coefficients of the DWT of the process $B_H(t)$ can be expressed as [17]:

$$\sigma^2_{d_{B_H j}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{B_H}\left(2^{-j}\nu\right) \cdot S_\psi(\nu) d\nu =$$
$$= \frac{c_f}{2\pi} 2^{j(2H+1)} \int_{-\infty}^{\infty} |\nu|^{-2H-1} S_\psi(\nu) d\nu = \quad (15)$$
$$= \frac{c_f I}{2\pi} 2^{j(2H+1)}, \; j = 1,..., J.$$

Hence, there is a recurrence relation between the variances of the wavelet coefficients at two consecutive scales:

$$\sigma^2_{d_{B_H j}} = \frac{c_f I}{2\pi} 2^{j(2H+1)} = 2^{2H+1} \sigma^2_{d_{B_H j-1}}, \quad (16)$$

This relation permits the estimation of the Hurst exponent as the slope of the line which represents the logarithm in base 2 of the variances $\sigma^2_{d_{B_{H_j}}}$ as a function of the decomposition level j, computed between two decomposition levels j_1 and j_2 [5, 16, 17]:

$$\log_2 \sigma^2_{d_{B_H j_2}} - \log_2 \sigma^2_{d_{B_H j_1}} = (2H+1)(j_2 - j_1). \quad (17)$$

Eq. (16) suggests the idea of the proposed wavelet-based fBm synthesis method (SM). The detail coefficients $d_{B_{H_{j,k}}}$ are first generated and next is applied the IDWT. The DWT of the fBm random process converges asymptotically to the Karhunen-Loève transform when $j \to \infty$:

$$\lim_{j\to\infty} \rho_{d_{B_H}}(k;j) = S_{B_H}(0)\delta(k), \quad (18)$$

So, as suggested by Wornell [1] and confirmed by Flandrin [5], a convenient simplification is to consider the detail coefficients $d_{B_{H_{k,j}}}$ as zero mean white Gaussian noises:

$$d_{B_{H_{k,j}}} = \frac{c_f I}{2\pi} 2^{j(2H+1)} d_{wn_{k,j}}\big|_{\sigma^2=1}, \quad (19)$$

for some scales $j_1 \le j \le j_2$. Accordingly, the equation (9) becomes:

$$B_H(t) = \frac{c_f I}{2\pi} \sum_{j=1}^{\infty} 2^{j(2H+1)} d_{wn\,k,j}\big|_{\sigma^2=1} \psi_{k,j}(t), \quad (20)$$

which represents the wavelet-based synthesis method for fBm processes SM. This synthesis method has two steps. The first step consists in the generation of the sequences of zero mean white Gaussian noise discrete in time processes with unitary variance $d_{wn_{k,j}}\big|_{\sigma^2=1}$ and in the modification of their variances which will become at scale j equal with $(c_f I / 2\pi) 2^{j(2H+1)}$. So, some sequences of white noises, indexed by j, with variances equal with the variances of the detail wavelet coefficients of fBm process from scale j but with different correlations are obtained. The second step of the proposed fBm synthesis method consists in the computation of the IDWT of the result obtained at the end of the first step. As opposed to the DWT, the IDWT will correlate the random process and will transform it in a non-stationary one because the DWT is perfectly invertible.

III. SIMULATION RESULTS

For our simulations we modified the DWT of white noise by choosing the variances of the sequences of the detail coefficients and the mean of the approximation coefficients. We have analyzed the precision of the generation of fBm processes with imposed value of the Hurst exponent.

The precision of the Hurst exponent obtained depends on the length of the input sequence, L, the mother wavelet used and on the desired value of the mean of the signal.

For each value of the Hurst exponent, from 0.1 to 1, going with a step of 0.1, we have generated 10000 sequences of fBm process of length L=16384 and we have estimated the value of the corresponding H with the aid of the MATLAB function `wfbmesti`. We have calculated the minimum and the maximum values of H and the standard deviations and we compared these values with the ones obtained using the MATLAB standard fBm generator. The function wfbm(H,L), the standard generator from MATLAB, returns a fractional Brownian motion signal fBm of Hurst exponent H (0 < H < 1) and length L, following the algorithm proposed by Abry and Sellan.

Table 1. Comparison between the values of Hmax and Hmin for the two generators

	New generator		wfBm generator	
	Hmin	Hmax	Hmin	Hmax
H=1	0.9990	1.0032	0.9581	1.0408
H=0.9	0.8592	0.9228	0.8615	0.9302
H=0.8	0.7530	0.8180	0.7576	0.8356
H=0.7	0.6368	0.7009	0.6571	0.7350
H=0.6	0.5643	0.6328	0.5594	0.6316
H=0.5	0.4899	0.5475	0.4999	0.5330
H=0.4	0.3823	0.4477	0.3785	0.4442
H=0.3	0.2782	0.3406	0.2896	0.3512
H=0.2	0.1648	0.2264	0.2067	0.2704
H=0.1	0.1070	0.1694	0.1393	0.2049

Table 2. Comparison between the values of the standard deviations for the two generators.

	New generator	wfBm generator
	Standard deviation	Standard deviation
H=1	0.0019	0.0127
H=0.9	0.0092	0.0121
H=0.8	0.0096	0.0118
H=0.7	0.0098	0.0119
H=0.6	0.0098	0.0114
H=0.5	0.0102	0.0109
H=0.4	0.0105	0.0108
H=0.3	0.0100	0.0109
H=0.2	0.0104	0.0105
H=0.1	0.0102	0.0101

From table 1, it can be seen that for most of the values of H, the value Hmin of our generator is bigger than the corresponding value for the standard generator from MATLAB, while the Hmax value is smaller for the proposed generator.

For almost all of the H values, the standard deviation is smaller for the new generator, as can be seen in table 2

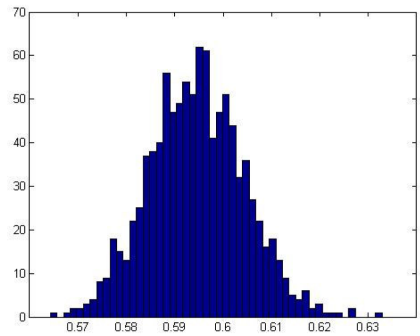

Fig.1 The histogram for H=0.6 of the proposed generator

We show also the two histograms for H=0.6

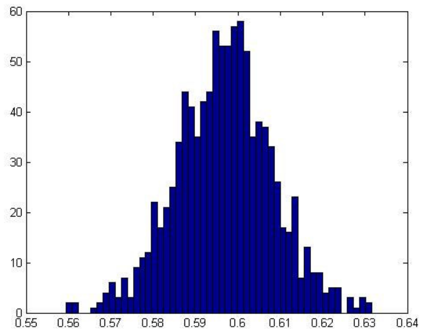

Fig.2 The histogram for H=0.6 of the MATLAB generator

IV. CONCLUSION

The aim of this paper is a new algorithm for the fBm processes generation. We have described the theoretical bases of the algorithm and its implementation. We have analyzed the performance of the proposed algorithm and we have proved that it is better than state of the art fBm generation algorithms.

REFERENCES

[1] G. W. Wornell, "A Karhunen-Loeve-like Expansion for 1/f Processes via Wavelets", *IEEE Transactions on Information Theory*, vol. 36, no. 4, July 1990, pp. 859-861.

[2] I. M. Gelfand, N. Ya. Vilenkin, *Generalized functions.* Vol. 4. *Applications of harmonic analysis.* Academic Press, New York,

[3] T. Blu and M. Unser, "Self-similarity: Part II – Optimal estimation of fractal processes*", IEEE Transactions on Signal Processing*, vol. 55, no. 4, April 2007, 1364-1378.

[4] B. B. Mandelbrot, J. W. Van Ness, "Fractional Brownian Motions, Fractional Noises and Applications", *SIAM Review*, vol. 10, no. 4, Oct. 1968, pp. 422-437.

[5] P. Flandrin, "Wavelets Analysis and Synthesis of Fractional Brownian Motion", *IEEE Transactions on Information Theory*, vol. 38, no.2, March 1992, pp. 910-917.

[6] W. Leland, M. Taqqu, W. Willinger, and D. Wilson, "On the self-similar nature of Ethernet traffic", *IEEE/ACM Trans.*

[7] J. Beran, R. Sherman, M. S. Taqqu, and W. Willinger, "Long range dependence in variable bit rate traffic," *IEEE Trans. Commun.*, vol. 43, nos. 2/3/4, pp. 1566–1579, 1995.

[8] V. Paxson and S. Floyd, "Wide-area traffic: The failure of Poisson modeling", in *Proc. SIGCOMM'94*, 1994.

[9] P. Flandrin, "On the Spectrum of Fractional Brownian Motions", *IEEE Transactions on Information Theory*, vol. 35, no. 1, January 1989, pp. 197-199.

[10] R. G. Clegg, "A practical guide to measuring the Hurst parameter", *Proceedings of 21st UK Performance Engineering Workshop*, School of Computing Science Technical Report Series, CS-TR-916, Univ. of Newcastle, 2005.

[11] P. Flandrin, *Representations temps-fréquence,* Hermes, 1993.

[12] S. Mallat, *A Wavelet Tour of Signal Processing*, 2nd Edition, Academic Press, New York, 1999.

[13] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, 1992.

[14] D. N. Veitch, Murad S. Taqqu and Patrice Abry, "Meaningful MRA initialisation for discrete time series", *Signal Processing*, vol. 80, Issue 9, September, 2000, 1971-1983.

[15] Isar, A., Nafornita, C., "On the Statistical Decorrelation of the 2D Discrete Wavelet Transform Coefficients of a Wide Sense Stationary Bivariate Random Process", *Digital Signal Processing*, Elsevier, vol. 24, January 2014, 95-105,

[16] P. Abry, D. Veitch, "Wavelet analysis of long range dependent traffic". *IEEE Trans. Inform. Theory*, vol. 44, no. 1,

[17] D. Isar, P. Gajitzki, "Pink noise generation using wavelets", *Proceedings of ISETc 2016,* Timi□oara, 2016.