FILTER BANK SYNTHESIS FOR ADAPTED WAVELET ANALYSIS

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Abstract: The multiresolution theory of orthogonal wavelets proves that any conjugate mirror filter characterizes a wavelet that generates an orthonormal basis of L^2R. This paper presents a method to obtain a discrete wavelet function, through FIR filter synthesis starting from a random discrete sequence. This sequence is used as a FIR filter which generates the four conjugate mirror filters necessary to implement the Discrete Wavelet Transform. The refinement of this sequence is carried out by a learning structure which follows to minimize the error between the signal and its reconstructed form from first order decomposition.

Keywords: *wavelet analysis, filter banks, perfect reconstruction.*

I. INTRODUCTION

The main idea in any signal decomposition is to represent the signal well, by a small number of basic functions. These can be sinusoids (Fourier analysis) or can be other functions (wavelets). The wavelet analysis is a new tool in signal processing and if the signal is represented as a function of time, wavelets provide efficient localization in both time and frequency domains. Important information often appears through a simultaneous analysis of the signal's time and frequency properties. This idea leads to decompositions over elementary atoms that are well concentrated in time and frequency. This is the main goal of the wavelet analysis, but the flexibility of time and frequency transforms is limited by the uncertainty principle, which states that the energy spread of a function and its Fourier transform cannot be simultaneously arbitrarily small.

For discrete-time signals, the Discrete Wavelet Transform (DWT) is equivalent to an octave filter bank, and can be implemented as a cascade of low-pass and high-pass finite impulse response (FIR) filters. The purpose of this paper is to find a filter bank for DWT implementation in order to have a signal-adapted decomposition.

II. WAVELETS AND FILTER BANKS

In signal analysis we want to represent a signal (a function) well, by a small number of basic signals. These functions are never periodic and we might hope that they can be represented as series expansion like [1]:

$$f(t) = \sum_{n=0}^{\infty} a_n f_n(t) \approx \sum_{n=0}^{N} a_n f_n(t) \tag{1}$$

The goal of wavelet analysis is to expand functions in terms of function of the type $\psi_{j,k}$ with f_n in (2) replaced by $\psi_{j,k}$, where $\psi_{j,k}, j,k \in Z$ is a family associated to function ψ defined by

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k), t \in R \tag{2}$$

The functions $\psi_{j,k}$ are scaled and translated versions of ψ (named the mother wavelet) and form the wavelet system associated to the function ψ. Wavelets are localized waves, instead of oscillating they converge fast to zero, they come from the iteration of rescaled filters [5]. Since we have two parameters in $\psi_{j,k}$ the expansion will be in terms of double sum. The purpose is to generate a set of expansion functions so that any discrete signal can be represented by the series

$$f(t) = \sum_{j,k} a_{j,k} \cdot \psi_{j.k}(t) \tag{3}$$

where the two-dimensional set of coefficients $a_{j,k}$ is called the *discrete wavelet transform* (DWT) of $f(t)$ [2]. A more exact form indicating how the $a_{j,k}$ coefficients are calculated can be written using inner products as

$$f(t) = \sum_{j,k} \langle \psi_{j.k}(t), f(t) \rangle \cdot \psi_{j.k}(t) \tag{4}$$

if the $\psi_{j,k}(t)$ form an orthonormal basis for the space of signals of interest. This expansion can only be possible under certain mathematical conditions on the function ψ. This wavelet series expansion is in terms of two indices, the time translation k and the scaling index j. This wavelet expansion provides a multiresolution analysis [1], which is connected in the signal processing domain with subband coding (or pyramidal) algorithm. This algorithm allows the implementation of discrete wavelet transform. The

Manuscript received September 2, 2010; revised October 31, 2010

multiresolution theory of orthogonal wavelets proves that any conjugate mirror filter characterizes a wavelet that generates an orthonormal basis of square integrable functions [5], [6], [7]. A time-scale representation of a discrete signal is obtained using digital filtering techniques. For discrete time signals the wavelet sequence is used to construct two finite impulse response (FIR) filters to decompose the signal into low and high frequency components as represented in figure 1.

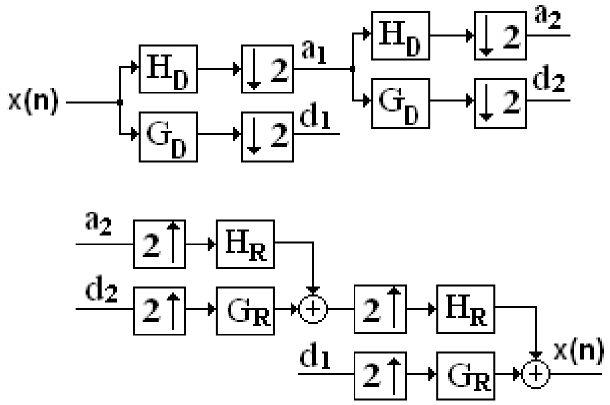

Figure 1. Filters for dyadic scale decomposition and reconstruction

The analysis bank is on the top and is composed from a low-pass filter H_D, a high-pass filter G_D and decimation which removes the odd numbered components after filtering. By analysis the input is separated into frequency bands.

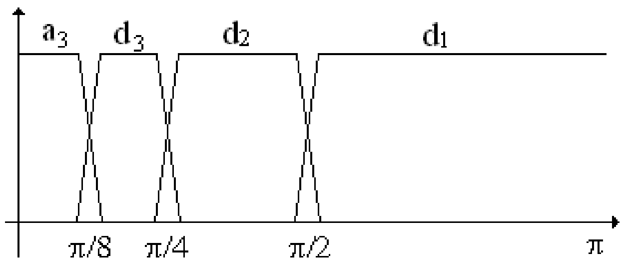

Figure 2. The obtained frequency bands

The resolution of the signal, which is a measure of the amount of detail information in the signal, is changed by the filtering operations and the scale is changed by upsampling and downsampling (subsampling) operations. Subsampling a signal corresponds to reducing the sampling rate, or removing some of the samples of the signal. The analysis structure yields two half-length outputs a_k and d_k. The operators H_D and G_D correspond to one stage in the wavelet decomposition, the spectrum of the signal is split in two equal parts, a low-pass (smoothed) and the high-pass part. The low-pass part can be split again and again until the number of bands created satisfy the computational demands. Thus, the discrete wavelet transformation can be summarised (after j stages) as

$$x \rightarrow \left(d_{j-1}, d_{j-2}, \ldots d_1, a_1\right) \quad (5)$$

where d_j are details and a_j average components. The synthesis bank (down on figure 1) begins with upsampling (which inserts zeros in odd components) and reassemble the signal. Upsampling a signal corresponds to increasing the sampling rate of a signal by adding new samples to the signal. The filters H_D, G_D, H_R, G_R are linear and time-invariant but the down- and upsampler operators are not time-invariant. These multirate operation can create extraneous signals which the filters must cancel. Generally, in filter banks there are two conditions for perfect reconstruction [6]. One condition removes distortion the other removes aliasing. The anti-distortion condition applies to the products H_RH_D and G_RG_D along the channels of the filter bank. The anti-aliasing condition controls how this products can be separated into four filters. The structure of an orthogonal filter bank (the length of the filter is 4 in this example and we suppose that the coefficients of the decomposition filter are; a, b, c, d) is very special, figure 3 shows how the filters (coefficients) are related.

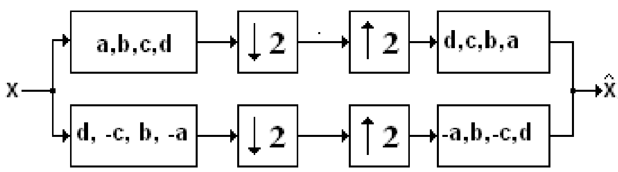

Figure 3. Orthogonal filter bank with four coefficients

A filter bank also gives perfect reconstruction if it is biorthogonal, in which case the design of the filter bank is less restrictive. One can say that the link between discrete-time filters and continuous-time wavelets is in the limit of the presented filter tree. Iteration of lowpass filters leads to the nscaling function [5].

III. THE PROPOSED METHOD

To obtain a well adapted wavelet function, a global error minimizing wavelet synthesizer method is proposed, the procedure to obtain the four FIR filters from the discrete wavelet sequence W is presented on figure 4.

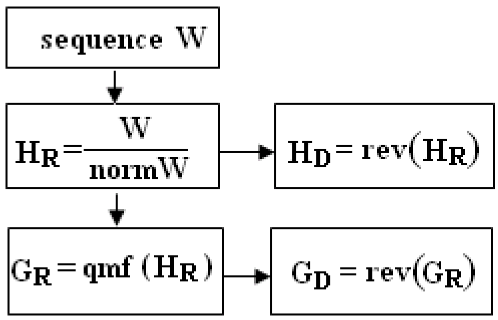

Figure 4. Obtaining decomposition and reconstruction filters from sequence W.

Starting from an arbitrary, discretized sequence, from which the reconstruction and decomposition filters can be obtained (figure 4), in each step a first order DWT is performed, the approximation signal is compared with the original one, the reconstruction error is computed, as illustrated in figure 4. According to the wavelet theory [5], [6] the filters must be low-pass FIR of 2N length, the sum of elements to be 1, with norm $\sqrt{2}$. The global error includes the errors from reconstruction and norm. The proposed algorithm to obtain a new wavelet sequence is presented in figure 5. We start with a random W sequence, the four filters are generated (according to figure 4) and the DWT of the test signal is performed with the starting sequence. In every step the error is calculated and the wavelet sequence is modified in order to reduce the error.

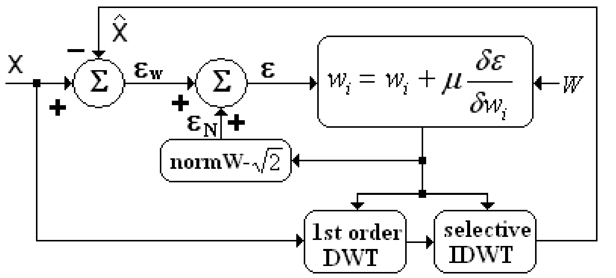

Figure 5. The proposed method to synthesize a discrete wavelet sequence

The error is defined as a sum of two errors,

$$\varepsilon = \varepsilon_w + \varepsilon_n \qquad (6)$$

where ε_w is a reconstruction (performed by Inverse Discrete Wavelet Transform) error (to measure the similarity between the signal and the obtained wavelet), and ε_n which shows how close the norm of the created wavelet is to the theoretical value. These are defined as:

$$\begin{aligned} \varepsilon_{wavelet} &= \{s - IDWT[TWD(s)]\}^2 \\ \varepsilon_{norm} &= \left(norm(w) - \sqrt{2}\right)^2 \end{aligned} \qquad (7)$$

The criterion-function to adjust the wavelet sequence was defined as:

$$w_i = w_i + \mu \frac{\delta\varepsilon}{\delta w_i} \qquad (8)$$

where μ is the learning rate and $\frac{\delta\varepsilon}{\delta w_i}$ is the variation of global error. The iteration stops when the error decreases under a certain, pre-fixed value or become constant.

IV. RESULTS

The used test signal is represented in figure 6. This is an artificially created signal which contains clean and noisy parts in the same structure, and it has a shape which is very close to an ECG signal. The length of the starting sequence was set to 8 (meaning fourth order filter), the resulting decomposition and reconstruction filters will have the same length. It is important to mention that when we start the decomposition, the used sequence is not a wavelet yet, but it is redefined after every step until it satisfy the mathematical conditions.

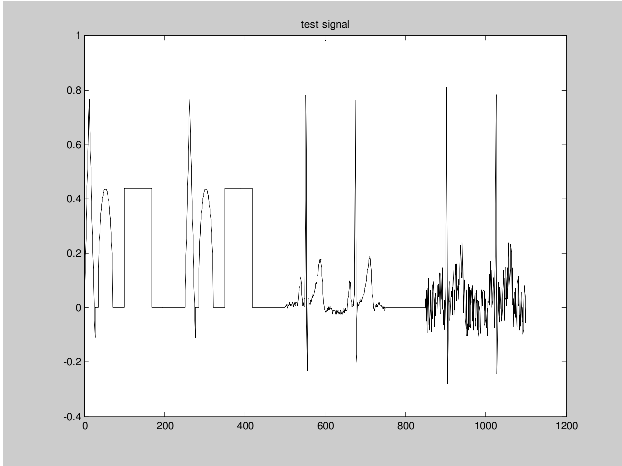

Figure 6. The test signal

The input signal, the reconstructed signal (only from approximation coefficients) and the difference between them are represented on figure 7. The wavelet sequence was modified in order to reduce this difference and to have certain mathematical conditions (as norm) satisfied. In this case the number of iteration was set to 2000, the learning rate was set to 0.004, the error's variation is represented in figure 8, the resulted norm for the new wavelet is 0.7076 (instead 0.7071, which is the theoretical value).

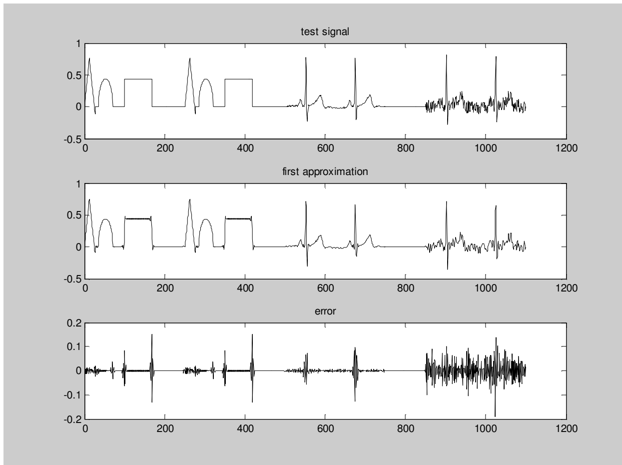

Figure 7. The input, reconstructed signals and the difference between them

The main task of the proposed algorithm is to minimize the reconstruction error. In this work is assumed that this error is the most important criterion to obtain a perfect reconstruction. For a number of 2000 iterations the evolution of the reconstruction error can be seen on figure 8. Increasing the number of iterations this error can be reduced.

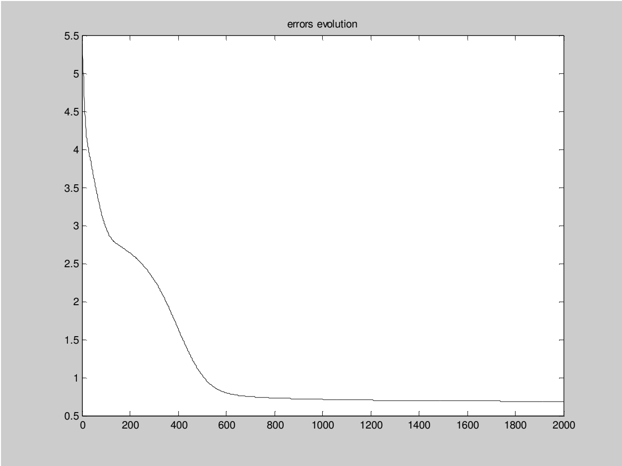

Figure 8. The evolution of the error (2000 iterations)

After a pre-fixed number of iterations the obtained wavelet sequence is presented on figure 8.

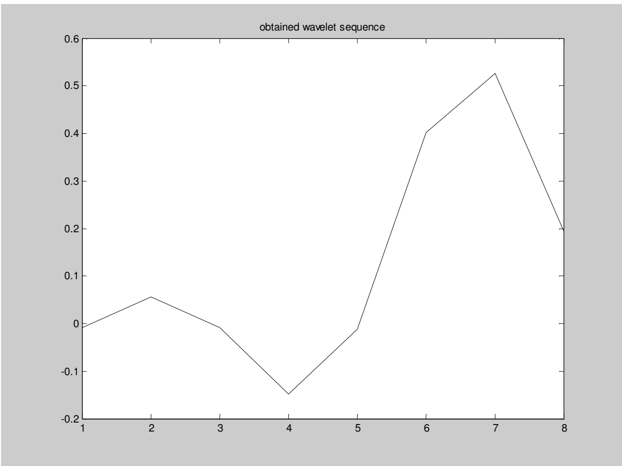

Figure 9. The resulted wavelet sequence

As we can see, its structure (shape) is very similar to other (already known and used in Matlab environment) wavelet functions. The four digital (decomposition and reconstruction low-pass and high-pass) filters obtained from the wavelet sequence are represented by coefficients in figure 10. One can see that the conditions for coeffiecients presented on figure 3 are totally satisfied, then the obtained sequence is a valid wavelet sequence.

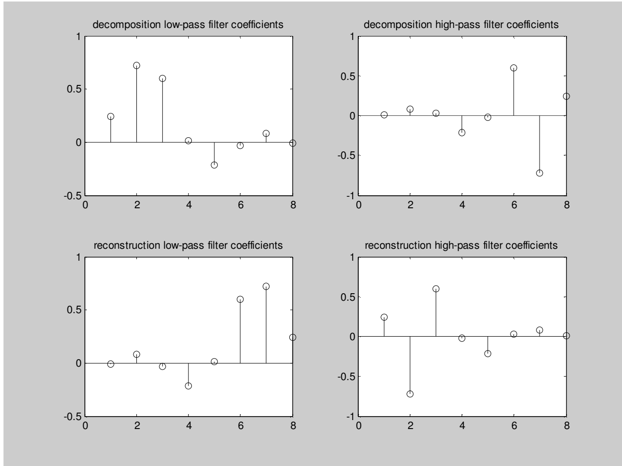

Figure 10. The obtained four

V. CONCLUSIONS

This papers shows that it is possible to obtain a discrete wavelet transform of a sequence that by using filter banks without specifying any function. The main advantage of the presented method is that the analyzing discrete wavelet is adapted to the signal. Every analyzed signal has its own discrete wavelet structure which perform the best reconstruction from first order approximation coefficients. The obtained functions gave almost the same results in decomposition, reconstruction as the existing functions. The original idea was to start from an arbitrary sequence, not to perform mathematical operations to obtain the wavelet sequence. One of disadvantages of the presented method is that the obtained sequence do not satisfy totally the mathematical conditions (the global error is non-zero), so for applications which require accurate information preservation (as compression) is not recommended. As further work the order of DWT can be increased, extra conditions for errors can be set and the length of the filters can be modified, depending on specified requirements. It is possible to obtain more adaptive sequences if only parts (of major interest) of signal are entered in the synthesizer.

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