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SIMULATION OF STOCHASTIC PROCESSES IN FINANCIAL MODELING

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Abstract: This paper discusses theoretical properties, shows the performance and presents some extensions of techniques used for simulation of stochastic differential equation applied on the financial data modeling. There are realized comparisons of different approaches for discretization schemes and their performances from the simulation convergence point of view. This study shows that, depending on the applicability of stochastic modeling to various financial data, the evolution of asset price over the time can be characterized by different processes accordingly with their dynamics.

Keywords: *stochastic differential equations, simulations schemes, jump diffusion, convergence schemes*

I. INTRODUCTION

Stochastic processes are one of the mathematical tools which are frequently used for modeling different phenomena in many fields as physics, biology, telecommunications, economics and financial mathematics [12]. One of the reasons of using stochastic processes as a mathematical tool for modeling the signals from these domains is related to the high degree of uncertainty in almost all phenomena. Since the deterministic component of a signal can be found by techniques like estimating the signal trend, filtering or Fourier analysis, the random components can be only estimated by modeling them a priori with stochastic processes.

Estimating the signals components involves an important amount of data, which in many fields, including the financial area, is difficult to obtain. In determining good estimators, the first step is to mathematically model and (then) to simulate the data [7]. The field of finance has been chosen to motivate the presented simulation methods. In this general framework, we focus our research for modeling the risk in finance [1], insurance and other economic areas. Thus, the simulations can conduct experiments under controlled conditions and they enable to determine what the effect of changing one factor or aspect of a problem will be, while leaving all others unchanged [11]. Often, the simulations models will express in mathematical equations a behavior of a dynamic system. Thus, simulations are particularly useful when models are very complex or the sample data sizes are small.

In practice is rather difficult to have very accurate estimations of the time series parameters and of inter-relationships between them. For example, there are some of the financial time series characteristics which can make the process of parameter estimation and inference less reliable. Some of these characteristics are the existence of fat tails, the structural breaks and bi-directional causality between dependent and independent variables. The fact that real data is unstructured can increase the degree of uncertainty of all of these features that lurk inside financial market data [2]. Clearly, it is important to have an idea of what the effects of such phenomena will be for inference and model estimation.

An important property of stochastic processes used in finance is that they can be modeled analytically with the stochastic differential equation (SDE) having the following expression [9]:

$$dX = f(X,t)dt + \sigma(X,t)dW \tag{1}$$

The main terms of equation are the drift function $f(X,t)$ and the diffusion (volatility) function $\sigma(X,t)$. The random component is given by the fundamental stochastic process, the Wiener process, $dW = Z\sqrt{dt}$ and Z is a normal variable: $Z \sim N(0,1)$. Depending on the application and on the numerical methods used in simulation, the drift and the diffusion functions may have various analytical forms.

Although stochastic differential equations are quite popular in finance and other domains, there is a lot of mathematics behind them. This fact involves different implementation approaches in software applications regardless to continuous and discrete form of the financial stochastic differential equation.

II. NUMERICAL METHODS IN STOCHASTIC PROCESES SIMULATION

In order to simulate and to make inference regarding the form of drift and diffusion function, different approaches and assumptions could be made, since there are more numerical schemes used for simulating stochastic differential equations. These methods are known as numerical solution of the stochastic differential equations.

Simulation methods are usually based on discrete

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approximations of the continuous solution of SDEs. The methods of approximation are classified according to their different properties [8]. Mainly two criteria of optimality are used in the literature: the strong and the weak (orders of) convergence.

The s*trong order of convergence* criterion is similar to the one used in the approximation of the trajectories of nonstochastic dynamical systems. A time-discretized approximation Y_δ of a continuous-time process *Y*, with δ the maximum time increment of the discretization [9], is said to be of general strong order of convergence γ if for any fixed time horizon *T,* it holds true that:

$$E\{\left|Y_\delta(T)-Y(T)\right|\}\le C\delta^\gamma,\ \forall\,\delta<\delta_0 \qquad (2)$$

with $\delta_0>0$ and *C* a constant not depending on δ. $E\{\}$ signifies the statistical average operator.

Along with the strong convergence, the *weak order of convergence* can be defined in a similar way. Y_δ is said to converge *weakly* of order β to *Y* if for any fixed horizon *T* and any $2(\beta+1)$ continuous differentiable function *g* of polynomial growth, it holds true that [9]:

$$\left|E\{g[Y(T)]\}-E\{g[Y_\delta(T)]\}\right|\le C\delta^\beta,\forall\delta<\delta_0,\delta_0>0 \quad (3)$$

Schemes of approximation of some order that strongly converge usually have a higher order of weak convergence.

Based on the order of convergence, there are two main approaches to numerical solutions of SDEs. The first one is based on numerical methods for ordinary differential equations [10]. The second one uses more information about the Wiener process [11]. From both categories, the most used approaches are the Euler-Maruyama (i.e. Euler) and the Milstein methods.

The most practical approximation scheme (method) is Euler's method. It has been widely used to generate solutions to deterministic differential equations by splitting up a time period into many small increments [8]. The number of increments will be sufficient when the model produces the same output for decision purposes as any greater number of increments.

Many of the current approaches [10] achieve a good approximation within a time interval by a Taylor series expansion which can be linear, or of higher order. As a consequence, the estimators of the model parameters will be affected by this discretization, so higher-order discretizations can be used as well to derive more precise results. Nevertheless, convergences issues could occur in regards with discretizations methods.

We are considering a one dimensional stochastic process $\{X_t,0\le t\le T\}$ which is the solution of the stochastic differential equation (1). Therefore, the expression (1) can be written as:

$$dX_t=f(X_t,t)dt+\sigma(X_t,t)dW_t \qquad (4)$$

with initial deterministic value $X_{t_0}=X_0$. Suppose that the stochastic process takes place in the time interval $[0,T]$. For two subsequent instants of time *t* and *r,* with (*r>t*), in a continuous time domain, the solution of the equation (4) in its stochastic integral form is:

$$X_t=X_r+\int_r^t f(X,\tau)d\tau+\int_r^t\sigma(X,\tau)dW \qquad (5)$$

In a discrete time the differential terms are transformed in difference terms: $\Delta t=(r-t)$ and $\Delta W=(W_r-W_t)$. Applying only a first-order (linear) approximation in a discrete time framework, the equation (5) can be written as follows:

$$X_r\cong X_t+f(X_t,t)\Delta t+\sigma(X_t,t)\Delta W \qquad (6)$$

where $f(X_t,t),\sigma(X_t,t)$ are the drift and the diffusion functions.

Generally speaking, the integration using Ito's calculus provides an estimate of the solution for the SDE based on a discretized scheme. Accordingly, this has the effect of introducing an uncertainty process which is not normal, thereby leading to stochastic volatility [8]. Thus, the Euler approximation of continuous stochastic process *X* is the *Y* discrete process satisfying the iterative scheme:

$$Y_{i+1}=Y_i+f(Y_i,t_i)dt+\sigma(Y_i,t_i)(W_{i+1}-W_i);i=1,N \quad (7)$$

The notation is simplified by setting $Y(t_i)=Y_i$, $W(t_i)=W_i$ and $Y_0=X_0$. *N* is the number of points used for discretization. Usually the time increment $\Delta t=t_{i+1}-t_i$ is taken to be constant:

$$h=\frac{T-t_0}{N}=t_{i+1}-t_i \qquad (8)$$

Between any two time points t_i and t_{i+1}, the process can be defined in various ways. One natural approach is to consider linear interpolation as the one from the first order Taylor expansion. In a similar way with the equation (6) the process *Y*(*t*) is defined as:

$$Y(t)=Y_i+\frac{t-t_i}{t_{i+1}-t_i}(Y_{i+1}-Y_i) \qquad (9)$$

In order to implement numerically the last equation and implicitly the equation (6), first we have to generate the random increments of the Wiener processes as independent normal random variables with $E\{\Delta W_t\}=0$ and $E\{(\Delta W_t)^2\}=h$. In a computer program, this can be easily obtained with the help of random number generators.

When a (Taylor) second-order approximation is taken, an additional source of uncertainty is added. In order to see how this occurs, it can be considered a more refined approximation based on a Taylor series expansion of the first two terms for the functions *f*() and σ(). This is used in the second discretization approach, the Milstein scheme. It makes use of Ito's lemma to increase the accuracy of the approximation by adding the second-order term only for the diffusion function [10]. Denoting by

σ_X the partial derivative of $\sigma(X,t)$ with respect to *X*, the Milstein approximation in an iterative form looks like:

$$X_{i+1} = X_i + f(X_t,t_i)(t_{i+1}-t_i) + \sigma(X_i,t_i)(W_{i+1}-W_i) + 0.5\sigma(X_t,t_i)\sigma_X(X_t,t_i)\{(W_{i+1}-W_i)^2 - (t_{i+1}-t_i)\} \quad (10)$$

or, in more symbolic form:

$$X_{i+1} = X_i + f\cdot\Delta t + \sigma\Delta W_t + \frac{1}{2}\sigma\sigma_X\{(\Delta W_t)^2 - \Delta t\} \quad (11)$$

Concerning the order of convergence, the Euler method is strongly convergent of order $\gamma = 0.5$ and weakly convergent of order $\beta = 1$ (under some smoothness conditions on the coefficients of the stochastic differential equation). The Milstein scheme converges with strong and weak order $\gamma = 1$, as stated in [14].

There can be also used an equivalence relationship between those two schemes. Given the generic stochastic differential equation (1), the Milstein scheme for it looks like:

$$\begin{aligned}\Delta X = X_{i+1} - X_i = &\\ (f(X_i,t_i) - 0.5\sigma(X_i,t_i)\sigma_x(X_i,t_i))\Delta t + &\\ + \sigma(X_i,t_i)\sqrt{\Delta t}Z + 0.5\sigma(X_i,t_i)\sigma_x(X_i,t_i)\Delta t Z^2&\end{aligned} \quad (12)$$

with $Z \sim N(0,1)$. We are considering the transformation $U = F(X)$ and its inverse $X = G(U)$. Then, according with Ito's lemma [6] the last expression can be written:

$$dU_t = \left(F'(X_t)f(X_t,t) + \frac{1}{2}F''(X_t)\sigma^2(X_t,t)\right)dt + + F'(X_t)\sigma(X_t,t)dW_t \quad (13)$$

with $U_t = F(X_t)$. If *F* is chosen as the Lamperti transform [11] so that $F'(X) = \frac{1}{\sigma(t,X)}$ and $F''(X) = -\frac{\sigma_X(t,X)}{\sigma^2(t,X)}$, then the final relation between the discretization schemes, as stated also in [10], becomes:

$$\begin{aligned}G(U_i + \Delta U) - G(U_i) = &\\ (f(X_i,t_i) - 0.5\sigma(X_i,t_i)\sigma_x(X_i,t_i))\Delta t + &\\ + \sigma(X_i,t_i)\sqrt{\Delta t}Z + &\\ + 0.5\sigma(X_i,t_i)\sigma_X(X_i,t_i)\Delta t Z^2 + O(\Delta t)^{\frac{3}{2}}&\end{aligned} \quad (14)$$

The last term $O(\Delta t)$ in the equation (14) is the remaining part from the Taylor expansion (higher than second order). Hence, the Milstein scheme on the original process and the Euler scheme on the transformed process are equal up to and including the order $O(\Delta t)$. In general, if *F* eliminates the interactions between the state of the process and the increments of the Wiener process, then this transformation method is probably always welcome because it reduces instability in the simulation process.

III. STOCHASTIC PROCESSES USED IN FINANCIAL MARKETS

In the real financial world the SDE are used to model different financial instruments prices or return evolutions across time. Depending on the analytical form of the drift and diffusion functions, there are several models used in practice.

If a price for an asset is considered to follow a random walk process [5] then the future price can be defined as:

$$X_{t+1} = X_t + N(\mu,\sigma) \quad (15)$$

The process change value in one unit of time is a quantity (a random number) that is normally distributed with mean μ and variance σ^2. Since in practice the assumption that the processes follow a normal distribution is not always true, it can be, at least for the beginning, a good choice from the computational point of view, to simulate processes having normal distribution. This assumption is valid especially when there are several samples for a process, whose distribution will converge to a normal one (according with Central Limit Theorem). In order to have a recurrence expression at different time intervals, the last equation can be iterated to obtain the relationship between X_t and X_{t+T} as:

$$X_{t+T} = X_t + N(\mu T, \sigma\sqrt{T}) \quad (16)$$

Thus, the last equation by dealing with discrete units of time has the advantage of being easy to implement in various computer programs. On the other hand, it can be written in a continuous time form considering any small time interval Δt:

$$\Delta X = N(\mu(\Delta t), \sigma\sqrt{\Delta t}) \quad (17)$$

Its equivalent SDE is:

$$dX = \mu dt + \sigma dW \quad (18)$$

The generalized Wiener process *dW* is sometimes called “perturbation”, “innovation” or “error” because is a gaussian (normal) white noise having the *N(0,1)* distribution.

The equation (15) allows the variable *X* to take any real value, including negative values. Since a lot of financial series like stock prices or interest rates do not take negative values, it won't be a good choice for modeling these kinds of time series. However, in order to make a prediction for some *X* value over a time interval *T* from now, only the value of *X* has to be known and nothing about the path how to get to the future value. The equation (18) is called the *Arithmetic Brownian Motion* (ABM) and it is used in other economics areas rather then in the real financial markets modeling.

Therefore, the return of a stock can be modeled as following expression, which has also its equivalent differential equation [2]:

$$r = \frac{dS}{S} = \mu dt + \sigma dW \Leftrightarrow dS = \mu S dt + \sigma S dW \quad (19)$$

The last equation is the *Geometric Brownian Motion* (GBM) process and is the simplest and frequently used stochastic process for modeling financial time series. Integrating the equation (19) involves the discretization schemes already presented. Considering that the general GBM's model equation is:

$$dX_t = \mu X_t dt + \sigma X_t dW_t \tag{20}$$

then the Euler discretization for this process looks like:

$$X_{i+1}^E = X_i^E\left(1+\mu\cdot\Delta t\right)+\sigma X_i^E \Delta W_t \tag{21}$$

and the Milstein scheme:

$$X_{i+1}^M = X_i^M + \mu X_i^M \Delta t + \sigma X_i^M \Delta W_t + \frac{1}{2}\sigma^2 X_i^M\{(\Delta W_t)^2 - \Delta t\} \tag{22}$$

Thus

$$X_{i+1}^E = X_i^E\left(1+\mu\cdot\Delta t+\sigma\sqrt{\Delta t}Z\right) \tag{23}$$

and

$$X_{i+1}^M = X_i^M\left\{1+\left[\mu+\frac{1}{2}\sigma^2(Z^2-1)\Delta t\right]+\sigma Z\sqrt{\Delta t}\right\} \tag{24}$$

Therefore, its associated Milstein scheme can be implemented by a Taylor expansion, which leads to:

$$\begin{aligned} X_{t+\Delta t} &= X_t \exp\left\{\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t+\sigma\sqrt{\Delta t}Z\right\} = Y_{i+1}^M \\ &= X_t\left\{1+\left(\mu-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}Z+\frac{1}{2}\sigma^2\Delta t Z^2\right\} \end{aligned} \tag{25}$$

This process can be easily transformed into another process in order to have equivalence between Milstein and Euler schemes. The *Itô's lemma* [6] applied to a function *F* of a variable *X*, which can be an asset price, is applied then to the return value of a stock. The Ito's stochastic calculus formula is*:*

$$dF = \frac{dF}{dX}dX + \frac{1}{2}\frac{d^2F}{dX^2}dt \tag{26}$$

Since the stock's return can be defined with the relation: $dX/X = d(\log[X])$, the return for an asset(stock price) can be rewritten, using $F(X) = \log[X]$ and the Euler scheme on the transformed process as:

$$\Delta \log X = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\,z \tag{27}$$

Now, using the Taylor expansion on the inverse transform $G(y) = e^y$, one can get the Milstein scheme.

The mean reversion process which will be presented in more details as follows is a modification to GBM that progressively encourages the series to move back towards a mean as the time horizon is increasing. Jump Diffusion, discussed after that, acknowledges that there may be shocks to the variable resulting in large discrete jumps.

Geometric Brownian motion with mean reversion is the stochastic process having as drift function the (complete) linear function. The long-run time-series properties of equity prices (amongst other variables) are, of particular interest to processing techniques applied on financial data. There is a strong interest in determining whether stock prices can be characterized as random walk or mean reverting processes because this could have an important effect on an asset's value. A stock price follows a mean reverting process if it has a tendency to return to some average value over time, which means that investors may be able to forecast future returns better by using information on past returns to determine the level of reversion to the long-term trend path. A random walk has no memory, which means that any large move in a stock price following a random walk process is permanent and there is no tendency for the price level to return to a trend path over time [5]. Increased volatility lowers a stock's value, so a reduction in volatility due to mean reversion would increase a stock's value.

For mean reversion process the ABM equation (18) can be modified as follows:

$$dX = \alpha(\mu - X)dt + \sigma dW \tag{28}$$

where $\alpha > 0$ is the speed of reversion. By being a mean reversion process, it tends to oscillate around some equilibrium state. Another interesting property of this process is that, contrary to the Brownian motion, it is a process with finite variance for all $t > 0$

The effect of the *dt* coefficient is to produce an expectation of moving downwards if *X* is currently above μ and vice versa. Mean reversion models are produced in terms of *S*(prices*)* or *r*(returns*)*. Thus, the underlying stochastic process X_t can describe either time series *S* or *r*.

The equation (28) is known as the *Ornstein-Uhlenbeck process* (or the Vasicek model) and it was one of the first models used to describe short term interest rates [16].

The *Cox-Ingersoll-Ross process* is a slight modification to the Ornstein-Uhlenbeck process**.** This process, also called CIR model [4], is used for modeling the interest rate on the short time horizon and also on the evolution of stock prices. It has the property of not taking negative values (so it can be used to model the variable *X* – the price of stocks) because the volatility goes to zero as *X* approaches zero:

$$dX = \alpha(\mu - X)dt + \sigma\sqrt{X}\,dW \tag{29}$$

Thus, this process can be simulated by using both already discussed discretization schemes. The associated Milstein scheme looks like:

$$\Delta X = \left((\alpha\mu - \alpha X_i) - \frac{1}{4}\sigma^2\right)\Delta t + \sigma\sqrt{X_i}\sqrt{\Delta t}Z + \frac{1}{4}\sigma^2\Delta t Z^2 \tag{30}$$

Now, using the transformation $y = \sqrt{x}$, the Euler scheme associated to the transformed SDE is:

$$\Delta Y = \frac{1}{2Y_i}\left(\left(\alpha\mu - \alpha Y_i^2\right) - \frac{1}{4}\sigma^2\right)\Delta t + \frac{1}{2}\sigma\sqrt{\Delta t}Z \qquad (31)$$

Since $G(y) = y^2$, the following transformation is obtained regardless to Milstein scheme:

$$G(Y_i + \Delta Y) - G(Y_i) = (Y_i + \Delta Y)^2 - Y_t^2 = (\Delta Y)^2 + 2Y\Delta Y_i =$$
$$\frac{1}{4}\sigma^2 \Delta t Z^2 + O(\Delta t^2) + \left(\left(\alpha\mu - \alpha Y_i^2\right) - \frac{1}{4}\alpha^2\right)\Delta t + Y_i\sigma\sqrt{\Delta t}Z \qquad (32)$$

As a consequence, the Euler method can also be used as a simulation scheme for this process.

The *stochastic process with jump diffusion* is another way of modeling the stock prices with geometric Brownian motion which provide the most flexible, numerically accessible, mathematical framework that is allowing modeling the evolution of financial and other random quantities over time [12]. In particular, feedback effects can be easily included and jumps are enabling the events modeling.

Jump diffusion refers to rare and sudden shocks in the prices that may occur randomly in time. The idea is to recognize that beyond the usual background randomness of a financial time series there will be events that have a much larger impact on the variable. Some rare events are considered to be the important changes in a company management, the announcement of last period inflation rate, change of government, etc. The frequency of these events and respectively the frequency of the jumps can be modeled as a Poisson process with intensity λ so that in a time frame T there will be $Poisson(\lambda T)$ jumps. By adding jump diffusion to the discrete GBM equation for one time period, its general expression becomes [3]:

$$dX = f(X,t)dt + \sigma(X,t)dW + q(X,t)dP \qquad (33)$$

In this equation the function $q(X,t)$ is the associated Poisson function to the stochastic process X and P_t is a simple Poisson process characterized by the λ parameter whose increments $\Delta P_t = P_{t+\Delta t} - P_t$ are given by:

$$\Delta P_t = \begin{cases} 1, & with\ probability\ \lambda\Delta t \\ 0, & with\ probability\ 1-\lambda\Delta t \end{cases} \qquad (34)$$

Geometrical Brownian motion with jump diffusion and mean reversion is a complex stochastic process which can be very well fitted for stock prices. In the case if the return r (for a stock price) has just received a large shock there might be a "correction" over time which brings the return evolution back to the expected return μ of the series. Combining mean reversion with jump diffusion will allow us to model these characteristics quite well and with few parameters. However, the GBM with jump diffusion model already presented, no longer applies for mean and variance, particularly when the reversion speed is large because one needs to model within the period the jump took place.

IV. SIMULATION RESULTS

This part of the paper is presenting some results regarding different stochastic processes and simulation schemes.

The most important and at the same time the most used process in the financial area from those presented is the GBM. In order to obtain the value $X_{t+\Delta t}$ starting from initial value X_t, one has to simply integrate over the time Δt the equation (19). The GBM model can be simulated also in terms of returns. By applying the Milstein scheme (24) which has a higher accuracy, we simulated more sample path of this process.

The spread of possible values in a GBM increases rapidly with time. For example, the following plot shows 20 possible forecasts with $X_0 = 1$, $\mu = 0.001$ and $\sigma = 0.02$:

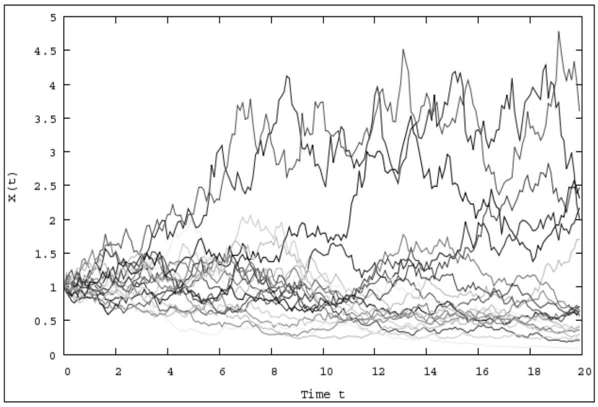

Figure 1: Plot of 20 possible paths with a GBM model described by (21).Its parameters are: μ = 0.01, σ = 0.2 and the starting value is 1.

In Figure 1 are plotted paths of GBM process which is simulated with Euler method, using a time increment of 0.01 and a number of observations in each dataset N = 1000.

These paths represent usually the model of stock market price evolution. On the other hand, the return r of a stock price denoted by $X=S$ is considered to be the logarithm of the fractional change in the stock's value as defined in equation (19). From the definition of a lognormal random variable, if $log[X]$ is normally distributed, then X is lognormally distributed. The equation for $X_{t+\Delta t}$ is modeling the stock prices as a lognormal random variable. Hence, the mean for X_{t+T} can be expressed as:

$$E\{X_{t+T}\} = X_t e^{\mu T} \qquad (35)$$

The drift μ is the exponential growth rate. The variance is given by:

$$V\{X_{t+T}\} = e^{2\mu T}(e^{\sigma^2 T} - 1) \qquad (36)$$

Hence, using this model and its associated properties, it is possible to make prediction about future price of a share also at timestamps with unequal time increments.

The size of time increment Δt and at the same time the number of observations has a strong impact on the quality of simulations as can been seen in the following experiments.

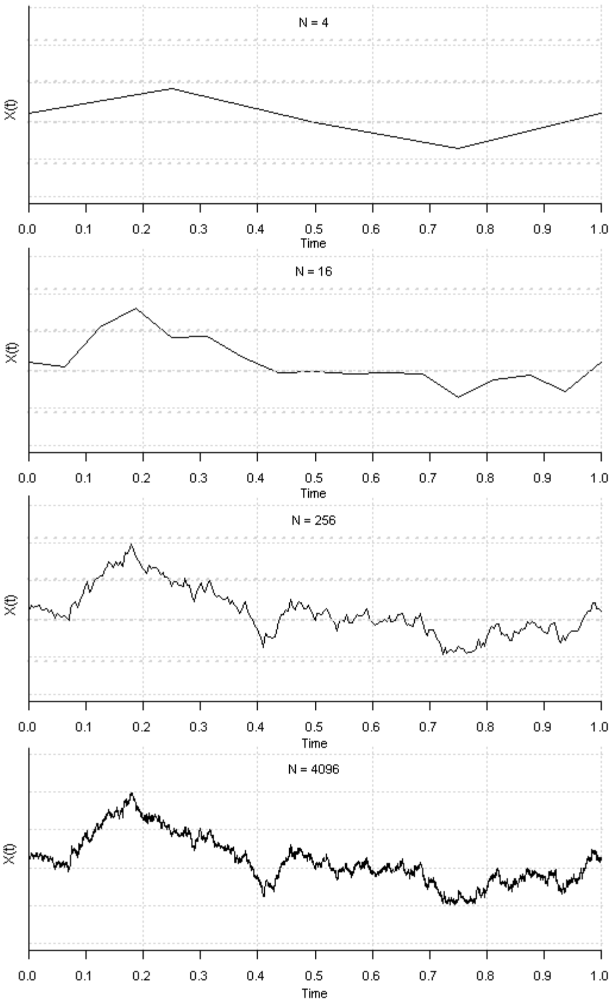

Figure 2: Influence on the simulation quality of number of observations (N).

If the time increment in the simulation is constant (Δt =0.1) and at the same time if we increase the number of observation for an SDE, the simulation in a discrete form is converging to the corresponding continuous form. In order to have a good approximation of a continuous stochastic process (in our case the GBM stochastic process from equation (19)) a number of observations N higher than 250 is sufficient. The form of the process is the same in all figures, due to the fact that the same random number generator was used with the same starting sequence.

One interesting experiment regarding the convergence of the presented schemes is to simulate the speed of convergence in respect to the number of observation in a sample path. As an example, we simulated the GBM process for different number of observations (e.g. N=256, N=204, etc.)

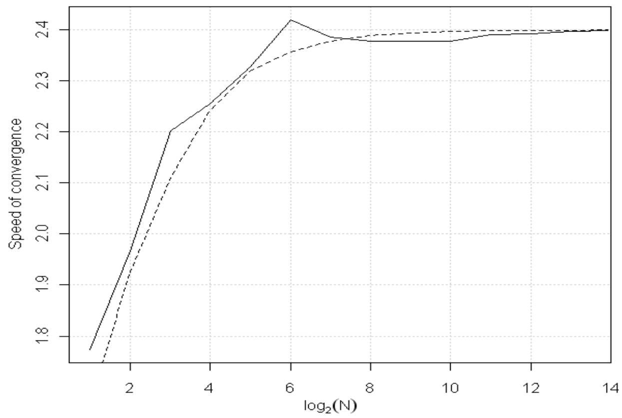

Figure 3: Speed of convergence for Euler and Milstein schemes.

Figure 3 shows the speed of convergence of both schemes (Euler with solid and Milstein with dashed line) to the true value (dot line at 2.4) as a function of Δt = 1/N. It can be seen that Euler scheme has a slower convergence speed, regardless with Milstein scheme but for a number of observations higher than 500, the results for both simulation schemes are almost the same.

There are differences in the simulations results when using the two described schemes (Euler and Milstein) especially in case when the diffusion function is nonlinear. The next plots are showing the simulation results for the Cox-Ingersoll process for both schemes (in the first figure is implemented the Euler scheme and in the second the Milstein simulation scheme).

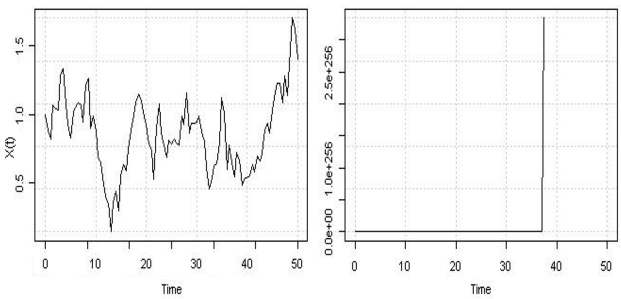

Figure 4: Euler method used for simulating the CIR process (in the left Δt=0.1 *and in the right* Δt = 0.5*).*

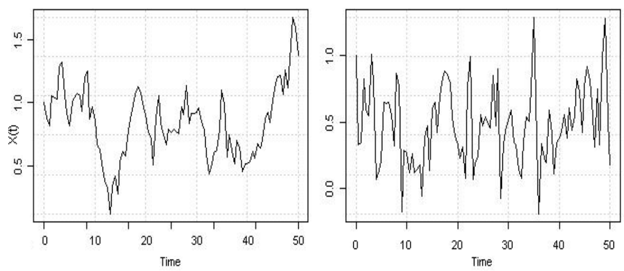

Figure 5: Milstein method used for simulating the CIR process (in the left Δt=0.1 *and in the right* Δt = 0.5*).*

From the previous figures it can be stated that in the case of CIR process, the Euler scheme has similar performance for a small increment of Δt <0.25 with the Milstein scheme. In case of higher values for Δt the Euler scheme does not converge.

The equation (28) which represents the *Ornstein-Uhlenbeck process,* without any constraints can give negative values if it is used for modeling stock prices. Using it in terms of returns, it will keep positive the values of X for any value of returns.

Since the solution of this equation has always positives values it can be used also for stock prices. Integrating over time the equation (28) in terms of returns, it gives the solution [14]:

$$r_{t+T} = N\left(\mu + e^{-\alpha T}(r_t - \mu), \sigma\sqrt{\frac{1-\exp(-2\alpha T)}{2\alpha}}\right) \quad (37)$$

In the next plot is shown some typical behavior for returns of assets modeled with Ornstein-Uhlenbeck process. From practice, as stated also in [13], typical values of α could be in the range of 0.1 to 0.3.

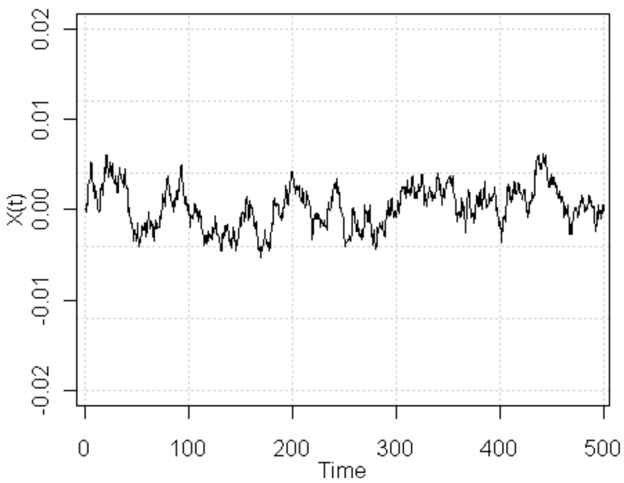

Figure 6. Samples of evolution of a stock returns modeled with Ornstein-Uhlenbeck process with parameters: μ = 0, σ = 0.001, α = 0.1

From the simulation point of view, the Ornstein-Uhlenbeck process (28) having $\sigma_X(X,t) = 0$. Therefore, this is one case for which, the both discretization schemes (i.e. Euler and Milstein), have the strong order of convergence $\gamma = 1$.

In the equation which is defining the *stochastic processes with jump diffusions* (33), the terms of type *q(X,t)dP* are defined as "jumps in the process". For the returns (r) of a stock price, the jump size is usually modeled as $N(\mu_J, \sigma_J)$ for mathematical and computational convenience.

The following two plots show a typical jump diffusion model giving both $X(t) = r(t)$ (return series) and $X(t) = S(t)$ (prices series).

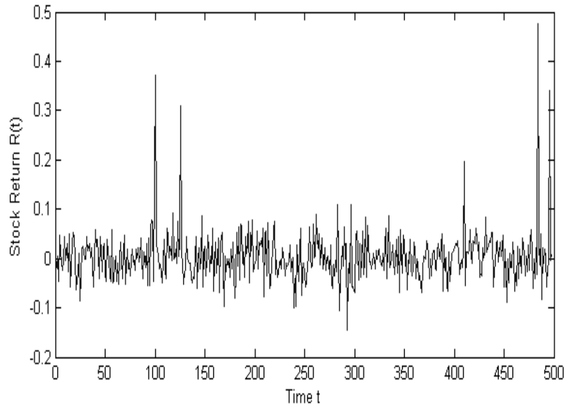

Figure 7. Samples of evolution of an asset return (stochastic process with Jump Diffusion) having the parameters: μ = 0, σ = 0.01, μ_J = 0.04, σ_J = 0.2 and λ = 0.02.

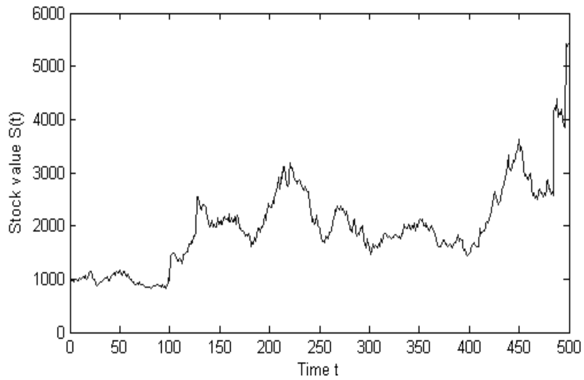

Figure 8. Samples of evolution of an asset price (stochastic process with Jump Diffusion) having the parameters: μ = 0, σ = 0.01, μ_J = 0.04, σ_J = 0.2 and λ = 0.02

The "jumps" terms from equation (33) are simulated as counting process as can be shown in the following expression [15]:

$$q(X,t)dP = \sum_{i=1}^{Poisson(\lambda)} N(\mu_J, \sigma_J) \quad (38)$$

For the simplicity k is defined as *Poisson(λt)* and at time t the equation (33) can be written as:

$$r_t = N\left(\left(\mu - \frac{\sigma^2}{2}\right)t + k\mu_J, \sqrt{\sigma^2 t + k\sigma_J^2}\right) \quad (39)$$

In the last equation the term k is generated with a Poisson random number generator.

The last two plots are presenting the simulations results for one of the most complex stochastic process used in financial modeling. Its complexity is due to the difficulty of parameters estimations. Both, simulations and estimation of the equation (33) can be performed also with Monte Carlo techniques [9].

V. CONCLUSIONS AND FUTURE WORK

Our results are showing several approaches and aspects related to simulation of stochastic processes used in financial modeling.

The attractive characteristics of these simulation are related to the fact that volatility updating structure permits analytical solutions to be generated for standard asset prices and thus the model allows a fast calibration to given market data. Also when using the Ito's lemma and the transformation relationship between Milstein and Euler scheme, the form of the stochastic process used to model the price dynamics allows the usage of non-lognormal probability distributions.

On the other hand, related to the accuracy of using simulations in order to model financial data, there are some drawbacks and possible questions. It might be computationally expensive to perform very accurate simulations, because the number of replications required to generate precise solutions may be very large. Even if the number of replications is very large, the simulations will not give a precise answer to the problem if some unrealistic assumptions have been made regardless to the data generating process. For example, in the context of option pricing, when using the GBM model, the option valuations obtained from a simulation will not be accurate if the simulation process has the assumption of normally distributed errors, while the actual underlying returns series is fat-tailed.

Depending on the (pseudo)random number generator or on the computing software, the simulation results are often hard to replicate, unless the experiment has been set up so that the sequence of random draws is known and it can be reconstructed. Since this is rarely done in practice, the results of a simulation study will be somewhat specific to the given investigation. In that case, a repeat of the experiment would involve different sets of random draws and therefore would be likely to yield different results, particularly if the number of replications is small.

Hence, the presented stochastic models are specific to different types of financial data and the associated simulations results are experiment-specific. In order to perform well across a large time interval of maturities (like in case of financial derivatives), further extensions of the models are necessary (such as time-dependent parameters).

Therefore, it is possible to use different simulation schemes according with the model assumption. Whenever possible, Euler scheme on the transformed process is recommended to be used. When using a Milstein scheme, one prerequisite is to have the derivatives of the drift and the diffusion functions. If these functions are simple functions of the process itself $X(t)$ their expression can be entered analytically in the simulation method. Otherwise, when these functions are given in an empirical form, another differentiating approach could be considered. Depending on the desired order of convergence, for some special applications other time discretization schemes having a higher order should be used. Nevertheless, the Milstein scheme has a strong order convergence value and it seems to have enough performance for various type of financial application [8].

As a conclusion, simulations are an extremely useful tool that can be applied to an enormous variety of problems. However, like all tools, it is not so easy to find if their results can be exploited with success especially on the real financial market where the future prices are hard to be predicted with a high accuracy.

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REFERENCES

[1] L. Bachelier *Théorie de la spéculation, thèse de mathématique*, Paris, 1900.

[2] Z.A. Bodie, K. Marcus &A. J. Marcus, *Finance Investments,* 5th Ed, McGraw-Hill, 2001.

[3] R. Cont, P. Tankov, *Financial Modeling with Jump Processes*, Chapman & Hall / CRC Press, 2004

[4] J.C. Cox, M. Rubinstein, *Options Markets*, Prentice Hall, 1985.

[5] E.F. Fama, "Random Walks In Stock Market Prices", *Financial Analysts*, 1965

[6] K. Ito, "Lectures on Stochastic Processes", *Lecture Notes*, Tata Institute of Fundamental Research, Bombay, India, 1961

[7] S. M. Kay, *Fundamentals of statistical Signal Processing, Volume 1: Estimation Theory*, Prentice Hall, 1993

[8] P.E. Kloden, E. Platen, *Numerical Solution of Stochastic Differential Equations,* Springer, 1992.

[9] P. Kloden, E. Platen, H. Schurz, *Numerical Solution of SDE through Computer Experiments*, Springer, Berlin, 2000

[10] G.N. Milstein, "Weak approximation of solutions of systems of stochastic differential equations", *Theory of Probability and Applications*, 1985.

[11] B. Oksendal, *Stochastic differential equations – An introduction with applications*, Springer-Verlag, New York, 1995.

[12] G. Papanicolaou, J.B. Keller, D. McLaughin, "Diffusions in random media", *Surveys in Applied Mathematics*, Plenum Press, New York, 205–255.

[13] K. Sigman, *Simulating normal (Gaussian) random variables with applications to simulating Brownian motion and geometric Brownian motion in one and two dimensions*, 2007.

[14] S.E. Shreve, *Stochastic Calculus for Finance II: Continuous-Time Models*, Springer, New York, 2004.

[15] P.F.M. Teunis, A.H. Havelaar, "The beta-Poisson Dose-Response Model Is Not a Single-Hit Model", *Risk Analysis* 20,513-520,2000.

[16] O. Vasicek, "An equilibrium characterization of the term structure", *J. Financial Econ.*, 5, 177–188, 1977.