Discrete Mathematics

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Lecture 2

The Z-transform

- The unit impulse signal concentrated at k is the signal defined by

$$\delta_k(n) = \begin{cases} 0, & n \neq k, \\ 1, & n = k. \end{cases}$$

- The unit step signal is the signal defined by

$$\sigma(n) = \begin{cases} 0, & n < 0, \\ 1, & n \geq 0. \end{cases}$$

Definition

Let $x \in S_+$. The Z–transform of x is given by

$$\mathcal{Z}\{x(n)\}(z) := X(z) = \sum_{n=0}^{\infty} \frac{x(n)}{z^n},$$

wherever the series is convergent.

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6) **(Scaling)** Let $x \in S_+$ and $a \in \mathbb{C}^*$. Then,
$$\mathcal{Z}\left\{a^n x(n)\right\}(z) = X\left(\frac{z}{a}\right).$$

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and in general, if $p \in \mathbb{N}^*$, then

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8) **The inverse of the** Z**-transform.** If z_k, $k = \overline{1,p}$ are all the singular points of $z^{n-1}X(z)$, then

$$x(n) = \sum_{k=1}^{p} Res\left(z^{n-1}X(z); z_k\right),$$

where $Res(f(z); z_0)$ denotes the residue of $f(z)$ in the point z_0.

Practice problems

1) Compute the following Z-transforms:
   a) $\mathcal{Z}\left\{n+1\right\}(z)$,
   b) $\mathcal{Z}\left\{\frac{1}{n+2}\right\}(z)$,
   c) $\mathcal{Z}\left\{\frac{1}{(n+1)(n+2)(n+3)}\right\}(z)$,
   d) If

$$y(n) = \sum_{k=0}^{n} \frac{2^k}{k+1},$$

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   find $Y(z)$.

2) For the given Z-transform, $X(z)$, compute the corresponding signal $x \in S_+$:
   a) $X(z)=\frac{z}{(z-1)(z-2)}$,
   b) $X(z)=ze^{1/z}$.

The Discrete Fourier Transform (DFT)

Consider the following

- Let $x \in S_+$ be a finite signal with values $\{x(0), ..., x(N-1)\}$;
- We will identify the signal x above with the vector $x \in \mathbb{C}^N$;
- The Z-transform applied to the finite signal x becomes

$$X(z) = \sum_{k=0}^{N-1} \frac{x(k)}{z^k};$$

- By taking now $z = \omega^{-i}$, where ω is the N-th root of unity,

$$\omega = e^{-\frac{2\pi}{N}j},$$

  we are led to the DFT of x.

Definition

The discrete Fourier transform (DFT) of the signal x is the finite signal F_x defined by

$$F_x(i) := f_i = \sum_{k=0}^{N-1} x(k)\omega^{ik}, \quad i = \overline{0, N-1}.$$

The Discrete Fourier Transform (DFT)

We can view the signal x as a vector in $\mathbb{C}^N$ and write $x = (x(0), ..., x(N-1))^T$. In a similar way its DFT, $F_x \in \mathbb{C}^N$ and $F_x = (f_0, ..., f_{N-1})^T$. The DFT F_x is a linear transform given by

$$F_x = Wx,$$

where W is an $N \times N$ matrix with complex entries, such that, if we start indexing the rows and columns of W from 0, we have $W(i,k) = \omega^{ik}$, $i, k = \overline{0, N-1}$,

$$W = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & \omega & \dots & \omega^{N-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^i & \dots & \omega^{i(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{N-1} & \dots & \omega^{(N-1)^2} \end{pmatrix}, \overline{W} = \begin{pmatrix} 1 & \dots & 1 & \dots & 1 \\ 1 & \dots & \omega^{-k} & \dots & \omega^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & \omega^{-k(N-1)} & \dots & \omega^{-(N-1)^2} \end{pmatrix}.$$

The Inverse DFT

Teoremă

The matrix W *is invertible and its inverse is given by:*

$$W^{-1} = \frac{1}{N}\overline{W}.$$

The signal x *can be recovered from its Fourier transform* F_x *by*

$$x = \frac{1}{N}\overline{W}F_x.$$

Remarks.

- The DFT is an invertible linear transform;

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Remarks.

- The DFT is an invertible linear transform;
- Assume that $x \in \mathbb{R}^N$ is a real signal and N is even. Then

$$F_x\left(\frac{N}{2} + r\right) = F_x\left(\frac{N}{2} - r\right), r = 0, ..., \frac{N}{2}.$$

The DFT of a convolution product

- The convolution product. Let $x, y \in \mathbb{C}^N$. Then $x * y \in \mathbb{C}^N$ is given by

$$(x * y)(n) = \sum_{k=0}^{N-1} x(k) y(n-k),\ \ n = \overline{0, N-1}.$$

For the DFT of the convolution product we have

$$F_{x*y}(k) = F_x(k) F_y(k), k = \overline{0, N-1}.$$

- The correlation product. Let $x, y \in \mathbb{C}^N$. Then $x \circ y \in \mathbb{C}^N$ is given by

$$(x \circ y)(n) = \frac{1}{N} \sum_{k=0}^{N-1} x(k) y(n+k),\ \ n = \overline{0, N-1}.$$

Let x and y be real signals. The DFT of the correlation is given by

$$F_{x \circ y}(k) = \frac{1}{N} \overline{F}_x(k) F_y(k), k = \overline{0, N-1}.$$

Teoremă (Parceval's identities)

Let $x, y \in \mathbb{R}^N$. Then

a) $\sum_{p=0}^{N-1} x_p y_p = \frac{1}{N} \sum_{k=0}^{N-1} \overline{F}_x(k) F_y(k);$

b) $\sum_{p=0}^{N-1} x_p^2 = \frac{1}{N} \sum_{k=0}^{N-1} |F_x(k)|^2.$