Numerical Methods

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ETTI 2017

Lecture 3

Functions of exponential order

Let I denote one of the intervals $(0, \infty)$ or $[0, \infty)$.

Definition

A function $f : I \to \mathbb{C}$ is called a function (continuous signal) of exponential order if there exist $M > 0$, $a > 0$ and $\alpha \in \mathbb{R}$ such that for all $t > a$,

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d) The function $f(t) = e^{t^2}$ is NOT a function of exponential order.

The Laplace transform

Definition (The Laplace transform)

The **Laplace transform** of f is the function $F : D \to \mathbb{C}$ defined by

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt, \; s \in D,$$

where $D \subseteq \mathbb{C}$ is the set for which the integral above is convergent.

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Theorem (Existence of the Laplace transform)

Let f be a piecewise continuous function of exponential order such that

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Exercise. Compute the Laplace transform of $\sigma(t)$, i.e., $\mathcal{L}\{\sigma(t)\}(s)$.

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f) The Laplace transform of the integral:

$$\mathcal{L}\left\{\int_0^t f(u)du\right\}(s) = \frac{1}{s}\mathcal{L}\{f\}(s).$$

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g) If $f(t)$ and $g(t) = \frac{f(t)}{t}$ are piecewise continuous functions of exponential order, then

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Theorem (The Laplace transform of the convolution product)

$$\mathcal{L}\{f * g\}(s) = F(s)G(s).$$