Numerical Methods

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Random Variables

Random variables

Definition (Random Variable)

Let $(\Omega, \mathcal{K}, P)$ be a probability space. A mapping $X : \Omega \to \mathbb{R}$ is called a **random variable** if

$$\{\omega \in \Omega \mid X(\omega) \leq a\} \in \mathcal{K}, \forall a \in \mathbb{R}.$$

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Definition (CDF)

The cumulative **distribution function (cdf)** of the random variable X is the mapping $F_X : \mathbb{R} \to [0, 1]$, defined by

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Definition (Identically distributed random vars)

We say that two random variables X and Y are identically distributed if $F_X = F_Y$.

Properties of the cdf

The **cdf** $F_X(x)$ has the following properties:

i) F_x is a non-decreasing function;

ii) $\lim_{x \to -\infty} F_X(x) = 0$, $\lim_{x \to \infty} F_X(x) = 1$;

iii) $F_X(x)$ is a right-continuous function;

iv) $P(a < X \leq b) = F_X(b) - F_X(a)$.

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Discrete random variables:

- The range is finite or countable;
- The discrete random vars are represented by an ordered series:

$$X : \begin{pmatrix} x_1 & x_2 & \dots & x_n & \dots \\ p_1 & p_2 & \dots & p_n & \dots \end{pmatrix},$$

where $x_1 < x_2 < \dots$ and $p_i = P(X = x_i)$. Note that $\sum_i p_i = 1$.

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Continuous random variables. In this case, there exists a **probability density function (pdf)** $\rho_X : \mathbb{R} \to \mathbb{R}_+$ such that

$$F_X(x) = \int_{-\infty}^{x} \rho_X(t)dt.$$

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Note that $\int_{-\infty}^{\infty} \rho_X(t)dt = 1$.

Examples of random variables

1) **The binomial random variable.** Let $p \in (0,1)$ and $n \in \mathbb{N}^*$. We say that X follows the binomial distribution with parameters n and p, $X \sim Binom(n,p)$ if X has the ordered series

$$X : \begin{pmatrix} 0 & 1 & \dots & n \\ p_0 & p_1 & \dots & p_n \end{pmatrix},$$

where $q = 1 - p$ and

$$p_k := P(X = k) = \binom{n}{k} p^k q^{n-k} = P_{n,k}.$$

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2) **The Poisson random variable.** Let $\lambda > 0$. We say that a random variable follows the Poisson distribution with parameter λ, $X \sim Poisson(\lambda)$, if X has the ordered series:

$$X : \begin{pmatrix} k \\ p_k \end{pmatrix}_{k \in \mathbb{N}},$$

if

$$p_k := P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}.$$

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Exercise. Determine the connection between the two distributions above, by computing

$$\lim_{n \to \infty, np = \lambda} P_{n,k}.$$

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4) **The Uniform (cont) distribution.** Let $a, b \in \mathbb{R}$ such that $a < b$. We say that the random variable X has the uniform distribution on $[a, b]$, $X \sim Uniform(a, b)$, if X has the **pdf**

$$\rho_X(x) = \begin{cases} 0, & x \notin [a, b] \\ \frac{1}{b-a}, & x \in [a, b]. \end{cases}$$

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5) **The normal distribution.** Let $\mu \in \mathbb{R}$ and $\sigma > 0$. We say that X follows the normal distribution with parameters μ and σ, $X \sim N(\mu, \sigma)$ if its **pdf** is given by

$$\rho_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \ x \in \mathbb{R}.$$

Examples of random variables/Random vectors

6) **The exponential random variable.** Let $\lambda > 0$. We say that X follows the Exponential distribution with parameter λ, $X \sim Exp(\lambda)$ if its **pdf** is given by

$$\rho_X(x) = \begin{cases} 0, & x < 0 \\ \lambda e^{-\lambda x}, & x \geq 0. \end{cases}$$

Random vectors. Let $(\Omega, \mathcal{K}, P)$ be a probability space and $X_1, ..., X_n$ be random variables over this space. The random vector $\mathbf{X} = (X_1, ..., X_n)$ takes values of the form $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$, $x_i \in Range(X_i)$ and has the **cdf** $F_{\mathbf{X}} : \mathbb{R}^n \to \mathbb{R}$, given by:

$$F_{\mathbf{X}}(\mathbf{x}) := F_{\mathbf{X}}(x_1, ..., x_n) = P(X_1 \leq x_1, ..., X_n \leq x_n).$$

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Two-dimensional discrete random vectors. Let $\mathbf{Z} = (X, Y)$ be a discrete two-dimensional (bivariate) random vector. Then the function $f : \mathbb{R}^2 \to [0, 1]$ defined by

$$f(x, y) = f_{X,Y}(x, y) = P(X = x, Y = y)$$

is called the **joint probability mass function** (joint **pmf**). In this context $f_X(x) = P(X = x)$ is called the **marginal pmf** of X and is given by:

$$f_X(x) = \sum_y f(x, y) \text{ and } f_Y(y) = \sum_x f(x, y).$$

Discrete random vectors

Example. Let $\mathbf{Z} = (X, Y)$ be the discrete random pdf with the joint **pmf** given by:

$$f(0,0) = \frac{1}{12},\ f(1,0) = \frac{5}{12},\ f(0,1) = f(1,1) = \frac{1}{4} \text{ and } f(x,y) = 0 \text{ otherwise.}$$

Consider also the discrete random vector $\mathbf{W} = (U, V)$ with the joint **pmf** $g(x, y)$,

$$g(0,0) = g(0,1) = \frac{1}{6},\ g(1,0) = g(1,1) = \frac{1}{3} \text{ and } g(x,y) = 0 \text{ otherwise.}$$

i) Compute the marginal **pmf**s for X, Y, and U, V.

ii) Compare $f_X(x)$ to $f_U(x)$ and $f_Y(y)$ to $f_V(y)$.

iii) Based on the calculations above answer the question: *"If the marginal* **pmfs** *are known can the joint* **pmf** *be obtained?"*

Continuous random vectors

Definition (Joint pdf)

A function $\rho : \mathbb{R}^2 \to \mathbb{R}$ is called a **joint probability density function** (joint **pdf**) of the *continuous* two-dimensional (bivariate) random vector $\mathbf{Z} = (X, Y)$ if for any $D \subseteq \mathbb{R}^2$, we have

$$P((X, Y) \in D) = \int\int_D \rho(x, y) dxdy.$$

We also use the notation $\rho_{X,Y}(x, y)$ or $\rho_Z(x, y)$ depending on the context.

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The marginal **pdf**s are defined by:

$$\rho_X(x) = \int_{-\infty}^{\infty} \rho(x, y) dy, \quad \rho_Y(y) = \int_{-\infty}^{\infty} \rho(x, y) dx.$$

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Remark. Any function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\rho(x, y) \geq 0, \forall (x, y) \in \mathbb{R}^2 \text{ and } \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x, y) dxdy = 1$$

is the joint **pdf** of some continuous bivariate random vector $\mathbf{Z} = (X, Y)$.

Continuous random vectors

The joint **cdf** of the bivariate random vector $\mathbf{Z} = (X, Y)$ is given by

$$F(x, y) = F_Z(x, y) := F_{X,Y}(x, y) = P(X \leq x, Y \leq y).$$

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$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} \rho(u, v) dvdu.$$

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$$\rho(x, y) = \frac{\partial^2 F}{\partial x \partial y}(x, y)$$

at continuity points of $f(x, y)$.

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Example. A pair of random variables (X, Y) has the joint pdf

$$\rho(x,y) = \frac{\alpha}{\pi^2(1+x^2)(1+y^2)}, \forall (x,y) \in \mathbb{R}^2.$$

a) Determine α and the joint **cdf** $F(x,y)$.
b) Find $P((X,Y) \in [0,1] \times [0,1]$.

Conditional pmfs and pdfs

Definition (Conditional pmf)

Let $\mathbf{Z} = (X, Y)$ be a discrete bivariate vector with **joint pmf** $f(x, y) := P(X = x, Y = y)$ and **marginal pmfs** $f_X(x)$ and $f_Y(y)$. For any x such that $f_X(x) > 0$, the **conditional pmf** of Y given that $X = x$ is the function of y, $f(y|x)$, defined by

$$f(y|x) = P(Y = y|X = x) = \frac{f(x, y)}{f_X(x)}.$$

In a similar way, one defines $f(x|y)$, i.e., the **conditional pmf** of X given $Y = y$.

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Definition (Conditional pdf)

Let $\mathbf{Z} = (X, Y)$ be a continuous bivariate vector with **joint pdf** $\rho(x, y)$ and **marginal pdfs** $\rho_X(x)$ and $\rho_Y(y)$. For any x such that $\rho_X(x) > 0$, the **conditional pmf** of Y given that $X = x$ is the function of y, $\rho(y|x)$, defined by

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Independent random variables

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Let $\mathbf{Z} = (X, Y)$ be a bivariate random vector with joint **pdf** or **pmf** $\rho(x, y)$ and the marginal **pdfs** $\rho_X(x)$ and $\rho_Y(y)$. We say that X and Y are **independent** if

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Example. Consider the random variables X, Y with joint **pmf**, $f(x, y)$ given by

$$f(10, 1) = f(20, 1) = f(20, 2) = \frac{1}{10}$$

$$f(10, 2) = f(10, 3) = \frac{1}{5}, f(20, 3) = \frac{3}{10} \text{ and } f(x, y) = 0, \text{ otherwise.}$$

Determine whether the random variables X and Y are independent.

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Theorem (Checking independence)

Let X, Y be two random variables with joint **pdf** *or* **pmf** $\rho_(x, y)$*. Then X and Y are independent* **if and only if** *there exist functions $g(x)$ and $h(y)$ such that*

$$\rho(x, y) = g(x)h(y), \ \forall (x, y).$$

Functions of random variables. Examples.

Example (Discrete Random Variable). Let X be the discrete random variable given by the ordered series:

$$X : \begin{pmatrix} -2 & -1 & 1 & 2 \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{6} & \frac{1}{4} \end{pmatrix}$$

Compute the ordered series for the following random variables:

a) $U = X^3$,

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Exercise. Let $X \sim Binom(n, p)$, where $n \in \mathbb{N}^*$ and $p \in (0, 1)$. Find the **pmf** of the random variable $Y = n - X$.

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Functions of two random variables. Let X and Y be two random variables such that the values of the random vector (X, Y) are in the domain $D \subseteq \mathbb{R}^2$ and let $f : D \to \mathbb{R}$. The **cdf** of $Z = f(X, Y)$ is given by $F_Z(z) = P(f(X, Y) \leq z)$.

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$$F_Z(z) = \int\int_{D_z} \rho_{X,Y}(x, y)dxdy \text{ where } D_z = \{(x, y) \in \mathbb{R}^2 : \ f(x, y) \leq z\}.$$

Expectation (Mean) of a random variable

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i) (**Discrete case.**) Let X be a discrete random variable given by the ordered series: $X : \begin{pmatrix} x_k \\ p_k \end{pmatrix}$. Then, the **expectation (mean)** of the random variable X, $E[X]$, is defined to be

$$E[X] = \sum_k x_k p_k, \text{ provided it exists.}$$

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ii) (**Continuous case.**) Let X be a continuous random variable with **pdf** $\rho_X(x)$. Then the **expectation (mean)** of the random variable X is defined to be

$$E[X] = \int_{-\infty}^{\infty} x \rho_X(x) dx, \text{ provided it exists.}$$

Expectation (Mean) of a random variable

Definition (Expectation of a random variable)

i) (**Discrete case.**) Let X be a discrete random variable given by the ordered series: $X : \begin{pmatrix} x_k \\ p_k \end{pmatrix}$. Then, the **expectation (mean)** of the random variable X, $E[X]$, is defined to be

$$E[X] = \sum_k x_k p_k, \text{ provided it exists.}$$

ii) (**Continuous case.**) Let X be a continuous random variable with **pdf** $\rho_X(x)$. Then the **expectation (mean)** of the random variable X is defined to be

$$E[X] = \int_{-\infty}^{\infty} x\rho_X(x)dx, \text{ provided it exists.}$$

Definition (Moments of a random variable)

Let X be a random variable, $a \in \mathbb{R}$ a fixed number and $k \in \mathbb{N}^*$. Then $E[(X-a)^k]$ is called the k-**th order moment of** X **about** a provided it exists. If $a = 0$, then $E[X^k]$ is called the k-**th order moment** and if $a := E[X]$, it is called the k-**th order central moment** of X.

Expectation and variance

Exercise. Compute the expectation $E[X]$ if

a) $X \sim Binom(n, p)$,

b) $X \sim N(\mu, \sigma)$.

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Theorem (Properties of the expectation)

Let X, Y be random variables with expectations $E[X]$ and $E[Y]$ respectively. Let $a, b \in \mathbb{R}$. Then the following properties hold:

i) $E[aX + b] = aE[X] + b$.

ii) $E[X + Y] = E[X] + E[Y]$.

iii) *If X and Y are* **independent** *random variables then $E[XY] = E[X]E[Y]$.*

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Definition (Variance and Standard Deviation)

Let X be a random variable with $\mu := E[X] < \infty$. Then the **variance** or **dispersion** of X is denoted by $VAR[X]$ and it is defined by

$$VAR[X] = E[(x - \mu)^2].$$

The number $\sigma := \sigma_X = \sqrt{VAR[X]}$ is called the **standard deviation** of X.

Variance. Moment generating function

Theorem

Let X be a random variable with expectation $E[X]$ and variance $VAR[X]$. The following properties hold:

i) *(**Computing the variance**) $VAR[X] = E[X^2] - (E[X])^2$.*

ii) *(**Variance of an affine transformation**) Let $a, b \in \mathbb{R}$ fixed constants. Then $VAR[aX + b] = a^2 VAR[X]$.*

iii) *(**The variance of a sum of independent random variables**) Let $X_1, ..., X_n$ be independent random variables. Then,*

$$VAR\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} VAR[X_i].$$

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Definition (Moment generating functions)

Let X be a random variable. The **moment generating function (mgf)** of X, denoted by $M_X(t)$ is

$$M_X(t) = E\left[e^t X\right],$$

provided that the expectation exists for some t in a neighborhood of 0.

Moment generating functions. Covariance, correlation

Theorem (Generating moments)

If the random variable X has the **mgf** $M_X(t)$, *then*

$$E[X^n] = M_X^{(n)}(0).$$

Definition (Covariance, Correlation coefficient)

Let X and Y be two random variables with means $\mu_X = \overline{X} = E[X]$ and $\mu_Y = \overline{Y} = E[Y]$. Then, the covariance of the random variables X and Y is given by:

$$\mathrm{COV}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)].$$

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The correlation coefficient $r(X, Y)$ of the two random variables X, Y is given by

$$r(X, Y) = \frac{\text{COV}(X, Y)}{\sigma_X \sigma_Y}.$$

Covariance and correlation

Theorem

Let X and Y be two random variables with means $\mu_X = E[X]$ and $\mu_Y = E[Y]$. Then the following properties hold:

i) $COV(X, Y) = E[XY] - E[X]E[Y]$.

ii) *If X, Y are independent random variables, then $COV(X, Y) = 0$.*

iii) $\sigma_X^2 := VAR(X) = COV(X, X)$.

iv) *If X and Y are independent, then $r(X, Y) = 0$.*

v) $-1 \leq r(X, Y) \leq 1$.

vi) *If $r(X, Y) = \pm 1$, then $X - \mu_X$ and $Y - \mu_Y$ are linearly dependent.*