Random Variables and Random Vectors
Functions of Random Variables
Numerical Characteristics of Random Variables

Last Updated: 14/01/2018, 19:00

1. Let X be a continuous random variable with the **cdf** given by:

$$F_X(x) = \frac{c}{1+e^{-x}},\ x \in \mathbb{R}.$$

   (a) Find c.

   (b) Find the **pdf** $\rho_X(x)$.

   (c) Find $P(X \in [0, ln2))$.

2. Let X be a continuous random variable with **pdf** $\rho(x)$ and **cdf** $F(x)$ satisfying $F(1) < 1$. Let

$$g(x) = \begin{cases} \frac{\rho(x)}{1-F(1)}, & x \geq 1 \\ 0 & x < 1. \end{cases}$$

   Prove that $g(x)$ is a **pdf**.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$, be the function given by:

$$f(x,y) = \begin{cases} \alpha x y^2, & (x,y) \in (0,1) \times (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Determine α such that $f(x,y)$ is the joint **pdf** of two random variables X and Y.

   (b) Determine the marginal **pdf**s $f_X(x)$ and $f_Y(y)$.

   (c) Determine $P(X + Y \geq 1)$.

4. Let X be the random variable with the **pdf**

$$\rho_X(x) = \begin{cases} 30x^2(1-x)^2, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

   Find the **pdf** of the random variable $Y = X^2$.

5. Let X and Y be two continuous independent random variables with pdfs $\rho_X(x)$ and $\rho_Y(y)$ respectively. Determine the pdf of the random variable $Z = X + Y$.

6. Let $\beta > 0$ be fixed and

$$\rho_X(x) = \frac{c}{\beta^3\sqrt{\pi}} x^2 e^{-x^2/\beta^2}, x \geq 0 \text{ and } \rho_X(x) = 0, x < 0.$$

   (a) Find c such that $\rho_X(x)$ is the **pdf** of the random variable X.

   (b) Find $E[X]$ and $VAR[X]$.

7. Let X be a random variable with a standard normal distribution, i.e., $X \sim N(0,1)$. Find the **pdf** of $Y = |X|$, $E[Y]$ and $VAR[Y]$.

8. Find the moment generating function **mgf** for each of the following random variables:

   (a) X is the continuous random variable uniformly distributed on $[0, c]$, with $c > 0$ (fixed).

   (b) X the discrete random variable

$$X : \begin{pmatrix} k \\ p_k \end{pmatrix}_{k \in \mathbb{N}},$$

   where

$$p_k := P(X = k) = \binom{r + k - 1}{k} p^r (1 - p)^k, k = 0, 1, ...$$

   Here $p \in (0, 1)$ and $r \in \mathbb{N}^*$.