Seminar 1 Graphs

Remark: All graphs are supposed to be simple if nothing else is specified.
Definitions:
1. A graph $K_n = (V, E)$, $(n = |V|)$ with the maximum number of edges is called **complete** (or, equivalently, every pair of vertices are joined by an edge).
2. A digraph $D = (V, A)$ is called **symmetric** if $(v_i, v_j) \in A \Longrightarrow (v_j, v_i) \in A$.
3. A digraph $D = (V, A)$ is called **antisymmetric** if $(v_i, v_j) \in A \Longrightarrow (v_j, v_i) \notin A$.
4. A digraph is called **tournament** if is antisymmetric and its underlying graph is complete.
5. If $G = (V, E)$ is a graph, its **complement** $\overline{G}$ is the graph whose vertex set is V and whose edges join those pairs of vertices which are not joined in G (actually $\overline{G} = (V, \overline{E})$, where $\overline{E}$ is the set of edges of K_n which are not in E).

Problems:
1. Is it possible that the following lists are the degrees of all vertices of a simple graph? If so, give a pictorial representation.
(i) 2,2,2,3 (ii) 1,2,2,3,4 (iii) 2,2,4,4,4 (iv) 1,2,3,4
2. Is it possible that in a group of 11 persons, each shakes hands with exactly 5 persons?
3. If G has n vertices and their degrees are $d_1, d_2, \dots, d_n$, what are the degrees of the vertices of $\overline{G}$?
4. For the graph in Figure 1, complete the following statements with *walk, trail, path.*
a) $w\,v\,y\,x\,v\,u\,r\,s$ is a .... of length ... between w and s
b) $v\,x\,v\,u\,r$ is a .... of length ... between v and r
c) $u\,v\,y\,x\,v\,w$ is a .... of length ... between u and w
d) $r\,u\,v\,w\,y$ is a .... of length ... between r and y

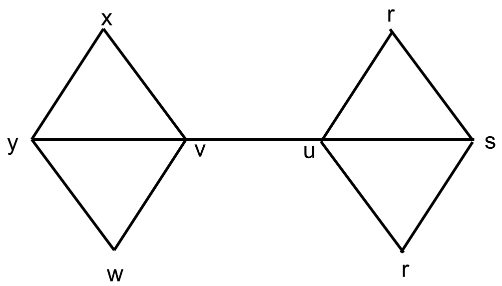

Figure 1:

5. a) For the graph in Figure 1, which edge is a bridge (cut-edge)? b) Write all the paths between `x` and `t`.
6. Find the number of edges in a complete graph.
7. In every graph $G = (V, E)$ with $|V| \geq 2$, there exist (at least) two vertices having the same degree.
8. Prove that a tree with n vertices, $n \geq 2$ has at least two vertices of degree 1.
9. If the degree of every vertex of a graph is at least 2, then the graph must contain a cycle.
10. Every walk W between two vertices x, y, $x \neq y$ is either a path or contains a closed sub-walk of W.

11. If there exist a (x, y)- walk in a graph G, then there exists also a (x, y)- path.
12. Every closed trail T contains a sub-walk that is a cycle. Show that this property is not true if T is merely a closed walk (by giving an example of a closed walk that does not contain a cycle).
13. Prove that if $F = (V, E)$ is a forest with c components, then $|E| = |V| - c$.
Definition: A collection of edge-disjoint cycles $C_1, C_2, \ldots, C_n$ is called a **decomposition** of the closed trail T if these cycles are sub-walks of T and if the union of their edge-sets coincides with the edge-set of T.
14. A closed trail can be decomposed into edge-disjoint cycles.