Seminar 6 (Homework)
Discrete probabilities

1. If X, Y are independent random variables, then

$$\text{var}(X \cdot Y) = \text{var}(X)\text{var}(Y) + E(X)^2\text{var}(Y) + E(Y)^2\text{var}(X).$$

2 For X, Y, Z random variable, prove the following equalities:

$$\text{var}(aX + bY) = a^2\text{var}(X) + b^2\text{var}(Y) + 2ab\ \text{cov}(X, Y),\ \forall\, a, b \in \mathbb{R},$$
$$\text{cov}(X + Y, Z) = \text{cov}(X, Z) + \text{cov}(Y, Z).$$

3. Suppose $X_1, X_2, \ldots, X_n$ are independent with $E(X_i) = \mu$ and $\text{var}(X_i) = \sigma^2$. Find $E((X_1 + X_2 + \ldots + X_n)^2)$.

4. A computer randomly generates passwords consisting only of numbers and (English) letters, distinguishing between uppercase and lowercase letters. If a password has 8 characters, what is the probability that it will contain 2 distinct lowercase letters, followed by 3 distinct uppercase letters and then 3 distinct digits?

5. Suppose that four new computer models M_1, M_2, M_3, M_4 are being tested for their reliability. The probability that a model satisfy the latest market standards are $p_1 = 0.8$ for model M_1, $p_1 = 0.7$ for model M_2, $p_3 = 0.9$ for model M_3 and $p_4 = 0.6$ for model M_4. Determine the probability p that at least three models match the profile.

6. It was found that the probability to log on to a computer from a remote terminal is 0.7. Let X denote the number of attempts that must be made to gain access to the computer.
a) Find the probability function of X. What type of distribution does X have?
b) Find the expected value of attempts needed to gain access to the computer.
c) Find the probability that at most 4 attempts must be made to gain access to the computer
d) Find the probability that at least 3 attempts must be made to gain access to the computer.

7. We are given two urns, the first containing 6 white balls and 4 black balls and the second 8 white balls and 6 black balls. We extract a ball from the first urn and introduce it into the second. Then we extract 4 balls from the second urn. Let X be the number of white balls from the 4 extracted from the second urn. Which is the expected value of X?

8. The telephone lines serving an information office are all busy about 60% of the times. Which is the expected number of tries for succeeding a reservation?

9. From an urn containing 8 white balls and 5 black balls we extract 3 times a ball, reintroducing it back each time. If X denote the number of white extracted balls, determine $E(X)$.

10. A multiple choice examination has 25 questions, each with 5 possible answers, only one of which is correct. Suppose that one of the students who takes the examination answers each of the question with an independent random guess.
a) Write the random variable representing the number of correct answers.
b) Which is the expected number of correct answers?

11. The number Y of customers per day at a certain sales counter has been observed for a long period of time and found to have a mean of 20 customers, with a standard deviation of 1 customer. The probability distribution of Y is not known. What can be said about the probability that Y will be between 16 and 24 tomorrow?

12. Let X and Y be independent random variables, each having a binomial distribution with parameters n, p and m, p, respectively, $p \in (0, 1)$. Which is the probability distribution of the random variable $X + Y$?

13. Eight letters are randomly distributed into 3 mailboxes. Let X be the number of letters in the 1^{st} mailbox. Find the probability function of X.
14. Let $X_1, \dots, X_n$ be independent and identically distributed random variables, with

$$X_i : \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}, \ i \in \mathbb{N}_n.$$

Find the probability distribution of $Y = \sum_{i=1}^n X_i$. What type of distribution does Y have?
15. Let $X_1, \dots, X_n$ be independent and identically distributed random variables, with a geometric distribution of parameter p

$$X_i : \begin{pmatrix} k \\ p\,q^{k-1} \end{pmatrix}_{k=1,2,\dots}.$$

Find the probability distribution of $Y = \sum_{i=1}^n X_i$. What type of distribution does Y have?
16. Suppose that the number of misprints per page in a book of 400 pages are independent and Poisson distributed with parameter λ. Determine the probability that: a) a page contains at least 2 misprints; b) the whole book has at least 2 misprints.
17. In an office n different letters are placed randomly into n addressed envelopes. Let Z_n denote the random variable that shows the number of correct mailings. For each $k \in \mathbb{N}_n$, let X_k be the random variable defined by

$$X_k = \begin{cases} 1, & \text{if the } k\text{-th letter is placed correctly,} \\ 0, & \text{otherwise.} \end{cases}$$

Calculate: a) $E(X_k)$ and $\operatorname{var}(X_k)$ for each $k \in \mathbb{N}_n$; b) $E(Z_n)$ and $\operatorname{var}(Z_n)$; c) $P(Z_n = i)$, for $i \in \{0, 1, 2, ..., n\}$.

Answers:
4. $24(26 \cdot 25)^2 \cdot 720/62^8$; **5.** Use Poisson's urns scheme. The required probability is $p = 0.7428$ **6.** a) Geometric distribution; b) 10/7 c) 0.9919 d) 0.09 **11.** Use Cebyshev inequality: $P(16 < Y < 24) \geq \frac{15}{16}$. **12.** $X+Y$ has a binomial distribution with parameters $m+n, p$. **13.** $P(X=i) = \frac{2^{8-i}\binom{8}{i}}{3^8}$, $i = \overline{0,8}$. **14.** Binomial with parameters n, p. **15.** Negative binomial distribution. **16.** The r.v. showing the nr of misprints on a page: $X : \begin{pmatrix} k \\ \frac{\lambda^k}{k!}e^{-\lambda} \end{pmatrix}_{k=0,1,\dots}$ The probability that a page contains at least 2 misprints is $\sum_{k=2}^\infty P(X=k) = 1 - e^{-\lambda} - \lambda e^{-\lambda}$. **17.** $P(X_k = 1) = \frac{(n-1)!}{n!} = \frac{1}{n}$, $E(X_k) = \frac{1}{n}$, $\operatorname{var}(X_k) = \frac{n-1}{n^2}$. b)$Z_n = X_1 + \ldots + X_n$, but the r.v. in this sum are not independent! We need $\operatorname{cov}(X_i, X_j) = \frac{1}{n^2(n-1)}$, so finally $\operatorname{var}(Z_n) = 1$.