Seminar 5
Discrete probabilities

1. Nine persons go into a 3-carriage tram. Each person chooses the carriage at random. Which are the probabilities of the following events: A : "Exactly three persons go into the first carriage", B : "There are three persons in each carriage", C : "There are 4 person in a carriage, 3 in another, and 2 in the other one"?

2. In studying the causes of power failures, the following data have been gathered: 10% are due to a transformer damage, 75% are due to line damage, 5% involve both problems. Based on these percentages, find the probability that a given power failure involves:

a) line damage given that there is transformer damage
b) transformer damage given that there is line damage
c) transformer damage but not line damage
d) transformer damage given that there is no line damage
e) transformer damage or line damage.

3. As communication security becomes more and more a problem, when a message is received, it must authenticated, by using a secret enciphering key. Sometimes, though, it can fall into wrong hands, thus allowing an unauthentic message to appear to be authentic. Assume that 80% of all messages received are authentic. Furthermore, assume that only 5% of all unauthentic messages are sent using the correct key and that all authentic messages are sent using the correct key. Find the probability that a message is authentic given that the correct key is used.

4. A computer center has three printers A,B and C, which print at different speeds. Programs are routed to the first available printer. The probability that a program is routed to printers A,B and C are 0.5, 0.3 and 0.2, respectively. Occasionally a printer will jam and destroy the printout. The probability that printers A,B and C will jam are 0.02, 0.06 and 0.1, respectively. Your program is destroyed by a printer jam. What is the probability that printer A is involved? Printer B is involved? Printer C is involved?

5. Three shooters shoot a target. The probability that they hit the target are 0.4, 0.5 and 0.7, respectively. Find the probability that the target is hit exactly once.

6. Let A,B be independent events. Show that:

a) A and $\overline{B}$ are independent
b) $\overline{A}$ and $\overline{B}$ are independent
c) if $A_1, A_2, \ldots, A_n$ are independent, then so are $\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_n$

7. If $P(B/\overline{A}) = P(B/A)$, then A and B are independent events.

8. In the game of bridge, each of the players gets 13 cards. Suppose North and South have 9 spades which include the A and the K but not the Q. Which event is more likely: A:"East has the Q" or B: "the four outstanding spades are split 2-2"?

9. (Coupon Collector's problem). Suppose that each box of cereal contains exactly one of r different coupons. Each coupon is equally likely to be placed in any given box and placing the coupons into the boxes are independent. Determine the probability that a person who buys n boxes of cereals ($r \leq n$) will obtain a complete set of r different coupons.

10. Suppose that n distinct balls are introduced randomly into r ($r \leq n$) distinct boxes in such a way that each of the n balls can go into any of the r boxes. Determine the probability that every box contains at least one ball.
11. In a study of plants near industrial waters, it was found that 30% showed signs of chemical pollution, 25% showed evidences of thermal pollution and 10% showed signs of both chemical and thermal pollution.
a) What is the probability that a stream that shows some thermal pollution will also show signs of chemical pollution?
b) What is the probability that a stream showing chemical pollution will not show signs of thermal pollution?
12. An urn contains n balls, numbered from 1 to n. We take the balls, one by one, without reintroducing them back. Find the probability that the numbers of the first k extracted balls coincide with the numbers of the extractions (ball number i extracted at the i-th extraction ($i = 1, 2, \ldots, k$).

Answers:
1. $P(A) = 7\left(\frac{2}{3}\right)^8$, $P(B) = \frac{35}{16}\left(\frac{2}{3}\right)^8$, $P(C) = \frac{280}{729}$. **2.** a) $\frac{1}{2}$ b) $\frac{1}{15}$ c) $\frac{1}{20}$ d) $\frac{1}{5}$ e) $\frac{4}{5}$ **3.** 0.987 **4.** $\frac{5}{24}, \frac{3}{8}, \frac{5}{12}$ **5.** 0.36 **6.** The events A, B are independent if $P(A) = P(A/B)$ $\left(P(A/B) = \frac{P(A \cap B)}{P(B)}\right)$ **8.** $P(A) = \frac{\binom{25}{12}}{\binom{26}{13}} = \frac{1}{2}$, $P(B) = \frac{\binom{22}{11}\binom{4}{2}}{\binom{26}{13}} = \frac{234}{575}$
9. Consider the events A_i: "the i^{th} coupon is missing from all n packages" and calculate $1 - P(A_1 \cup \ldots \cup A_r)$ using the sieve principle. We have $P(A_i) = \left(\frac{r-1}{r}\right)^n$, $P(A_i \cap A_j) = \left(\frac{r-2}{r}\right)^n$ etc. The required probability will be

$$1 - \sum_{j=1}^{r-1} (-1)^{j-1} \binom{r}{j} \left(1 - \frac{j}{r}\right)^n$$

11. a) $\frac{2}{5}$ b) $\frac{2}{3}$ **12.** $\frac{(n-k)!}{n!}$