Infinite series for computation of the probability density function of a sum of Rayleigh random variables

- The problem of determining the distribution function, or equivalently the complementary distribution function, of a sum of independent random variables, each of which possesses a Rayleigh distribution function has been of interest for the past 90 years.

- A convergent infinite series may be used for the computation of the complementary probability distribution function (c.d.f.) of a sum of independent Rayleigh random variables.

- Let $X_i, i = 1,...,L$ be bounded independent random variables each with p.d.f. $f_{X_i}(x_i)$
  - Conditions, notations:
    - $f_{X_i}(x_i) = 0$ for all $x_i < B_i^L$ or $x_i > B_i^U$.
    - Let be X the sum of the L random variables: $X = \sum_{i=1}^{L} X_i$, $G_X(x)$ the c.d.f. X and $f_X(x)$ the p.d.f. of X
    - X is lower bounded by $B_L = \sum_{i=1}^{L} B_i^L$ and upper bounded by $B_U = \sum_{i=1}^{L} B_i^U$.
    - Let be S(x) the periodic square wave given by:

$$S(x) = \begin{cases} 0, & -T/2 < x < 0 \\ 1, & 0 < x < T/2 \\ 1/2, & x = \pm T/2 \end{cases} \qquad (1)$$

$$S(x + mT) = S(x), m = 0, \pm 1, \pm 2, ... \qquad (2)$$

    - Then: $G_X(\varepsilon L) = \Pr(X > \varepsilon L) = E[S(X - \varepsilon L)]$ (3),

      where E[X] denotes the expected value of X and this equation is true for:

$$T/2 = \max\{B_U - \varepsilon L, \varepsilon L - B_L\} \qquad (4)$$

    - The Fourier series representation of S(x) is:

$$S(x) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \text{ impar}}}^{\infty} C_n e^{jn\omega x}, \quad C_n = \frac{1}{\pi n j}, \text{ where } \omega = \frac{2\pi}{T} \qquad (5).$$

- The combination of the above equations gives:

$$
\begin{aligned}
G_X(\varepsilon L) &= \frac{1}{2} + \sum_{\substack{n=-\infty \\ n\ impar}}^{\infty} C_n E[e^{jn\omega(X-\varepsilon L)}] \\
&= \frac{1}{2} + \sum_{\substack{n=1 \\ n\ impar}}^{\infty} \frac{E\left[e^{jn\omega(X-\varepsilon L)}\right] - E\left[e^{-jn\omega(X-\varepsilon L)}\right]}{n\pi j} \\
&= \frac{1}{2} + \sum_{\substack{n=1 \\ n\ impar}}^{\infty} \frac{E\left[\exp\left(jn\omega\left(\sum_{i=1}^{L} X_i - \varepsilon L\right)\right)\right] - E\left[\exp\left(-jn\omega\left(\sum_{i=1}^{L} X_i - \varepsilon L\right)\right)\right]}{n\pi j} \\
&= \frac{1}{2} + \sum_{\substack{n=1 \\ n\ impar}}^{\infty} \frac{\prod_{i=1}^{L} E\left[e^{jn\omega(X_i-\varepsilon)}\right] - \prod_{i=1}^{L} E\left[e^{-jn\omega(X_i-\varepsilon)}\right]}{n\pi j} \\
&= \frac{1}{2} + \sum_{\substack{n=1 \\ n\ impar}}^{\infty} \frac{\prod_{i=1}^{L} A_{in} e^{j\theta_{in}} - \prod_{i=1}^{L} A_{in} e^{-j\theta_{in}}}{n\pi j} \\
&= \frac{1}{2} + \sum_{\substack{n=1 \\ n\ impar}}^{\infty} \frac{\exp\left(j\sum_{i=1}^{L}\theta_{in}\right)\prod_{i=1}^{L} A_{in} - \exp\left(-j\sum_{i=1}^{L}\theta_{in}\right)\prod_{i=1}^{L} A_{in}}{n\pi j} \\
&= \frac{1}{2} + \sum_{\substack{n=1 \\ n\ impar}}^{\infty} \frac{A_n e^{j\theta_n} - A_n e^{-j\theta_n}}{n\pi j} \\
&= \frac{1}{2} + \frac{2}{\pi} \sum_{\substack{n=1 \\ n\ impar}}^{\infty} \frac{A_n \sin\theta_n}{n}
\end{aligned}
\qquad (6)
$$

- The Rayleigh p.d.f.. is:

$$
f_{X_i}(x_i) = \begin{cases} \frac{x_i}{\sigma_i^2} e^{-x^2/2\sigma_i^2}, & x_i \geq 0 \\ 0, & in\ rest \end{cases} \qquad (7),
$$

where $(2-\pi/2)\sigma_i^2$ is the variance of the distribution.

- The previously specified expectations :

$$
\begin{aligned}
E\left[\cos(n\omega X_i)\right] &= \int_0^\infty \frac{x_i}{\sigma_i^2} e^{-x_i^2/2\sigma_i^2} \cos(n\omega x_i)\, dx_i \\
&= \int_0^\infty u e^{-u^2/2} \cos(n\omega\sigma_i u) du = F\left(1, \frac{1}{2}, \frac{-n^2\omega^2\sigma_i^2}{2}\right)
\end{aligned}
\qquad (8),
$$

$$
\begin{aligned}
E\left[\sin(n\omega X_i)\right] &= \int_0^\infty \frac{x_i}{\sigma_i^2} e^{-x_i^2/2\sigma_i^2} \sin(n\omega x_i)\, dx_i \\
&= \sqrt{\frac{\pi}{2}} n\omega\sigma_i F\left(\frac{3}{2}, \frac{3}{2}, \frac{-n^2\omega^2\sigma_i^2}{2}\right) = \sqrt{\frac{\pi}{2}} n\omega\sigma_i e^{-n^2\omega^2\sigma_i^2/2}
\end{aligned}
\qquad (9)
$$

where F(.,.,.) is the hypergeometric function.

- Combining these two equations gives:

$$A_{in}=\sqrt{\left[F\left(1,\frac{1}{2},\frac{-n^2\omega^2\sigma_i^2}{2}\right)\right]^2+\frac{\pi}{2}n^2\omega^2\sigma_i^2e^{-n^2\omega^2\sigma_i^2}}$$

$$\theta_{in}=\tan^{-1}\left\{\frac{\sqrt{\frac{\pi}{2}}n\omega\sigma_i^2e^{-n^2\omega^2\sigma_i^2/2}\cos(n\omega\varepsilon)-F\left(1,\frac{1}{2},\frac{-n^2\omega^2\sigma_i^2}{2}\right)\sin(n\omega\varepsilon)}{F\left(1,\frac{1}{2},\frac{-n^2\omega^2\sigma_i^2}{2}\right)\cos(n\omega\varepsilon)+\sqrt{\frac{\pi}{2}}n\omega\sigma_i^2e^{-n^2\omega^2\sigma_i^2/2}\sin(n\omega\varepsilon)}\right\} \quad (10)$$

- Based on the Kummer transformation we have:

$$F\left(1,\frac{1}{2},-a\right)=e^{-a}F\left(-\frac{1}{2},\frac{1}{2},a\right) \quad (11)$$

for which the confluent hypergeometric function on the right side has infinite series expansion:

$$F\left(-\frac{1}{2},\frac{1}{2},a\right)=1+\frac{\left(-\frac{1}{2}\right)a}{\left(\frac{1}{2}\right)}+\frac{\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)a^2}{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)2!}+\frac{\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)a^3}{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)3!}+\ldots \quad (12)$$

$$=1-a-\frac{a^2}{3\cdot 2!}-\frac{a^3}{5\cdot 3!}-\ldots$$

Remark: This series converges much more rapidly than one obtained without applying the Kummer transformation.

- The truncation error when N terms are employed is given by:

$$R_N=-\sum_{i=N+1}^{\infty}\frac{a^i}{(2i-1)i!} \quad (13), \text{ and}$$

$$|R_N|=\sum_{i=N+1}^{\infty}\frac{a^i}{(2i-1)i!}<\frac{1}{2N+1}\frac{a^{N+1}}{(N+1)!}\sum_{i=0}^{\infty}\left(\frac{a}{N+2}\right)^i$$
$$=\frac{a^{N+1}(N+2)}{(2N+1)(N+1)!(N+2-a)}, \text{ for } a<N+2 \quad (14)$$

- Therefore, we have:

$$F\left(1,\frac{1}{2},-a\right)=-e^{-a}\sum_{i=0}^{\infty}\frac{a^i}{(2i-1)i!} \quad (15)$$

and the magnitude of the truncation error is upper bounded by:

$$|error|<e^{-a}\frac{a^{N+1}(N+2)}{(2N+1)(N+1)!(N+2-a)} \quad (16)$$

- The p.d.f. can be computed based on the c.d.f. – it is the derivative of the c.d.f. according to variable x:

$$f_X(x)=\lim_{T\to\infty}\frac{4}{T}\cdot\sum_{\substack{n=1\\ n\,odd}}^{\infty}A_n\cdot\cos(\theta_n) \quad (17)$$

Computation of the global p.d.f.

- The computation of the p.d.f. of the the amplitude of a signal affected by multipath propagation and Rayleigh fading may be done by decomposing the signal on two axes, sine and cosine, computation of the p.d.f. of the signal on both axes and then by the combination of the two p.d.f. – see fig. 1.
  - If we have a multipath propagation profile with N different paths, characterized by the attenuation of the paths, a_i, and the delays of the paths, τ_i, and if each path has fading Rayleigh, than the received signal will be:

$$s_r(t)=\sum_{i=0}^{N-1}F\left(A_t\cdot a_i/\sqrt{2}\right)\cdot\cos\left(\omega_c\cdot(t-\tau_i)\right)=\sum_{i=0}^{N-1}F\left(A_t\cdot a_i/\sqrt{2}\right)\cdot\cos\left(\omega_c\cdot t-\varphi_i\right)=$$
$$=\cos\left(\omega_c\cdot t\right)\cdot\sum_{i=0}^{N-1}F\left(A_t\cdot a_i\cdot\cos(\varphi_i)/\sqrt{2}\right)+\sin\left(\omega_c\cdot t\right)\cdot\sum_{i=0}^{N-1}F\left(A_t\cdot a_i\cdot\sin(\varphi_i)/\sqrt{2}\right) \quad (18)$$

  - It is considered that the Rayleigh fading does not affect the phase of the signal, only the amplitude of the signal with multipath propagation.
  - On both axes (I and Q) we have a sum of Rayleigh distributed variables; the finite series method is used to compute the p.d.f. of the signals on both axes, signals denoted as X_{cos} and X_{sin}.
  - The p.d.f. is calculated only in points εL, which means that it is obtained a sampled function, with a sampling step ΔL.
  - The sums of the random variables X_{cos} and X_{sin}, is random variable, X_v, with p.d.f. $f_V(x)$.
  - The probability that the variable X_v is in a square of area $(\Delta L)^2$, square delimited by [eL , (e+Δ)L] on the cosine axe and [dL , (d+Δ)L] on the sine axe
    - if step ΔL is sufficiently small, the probability that variable X_v is located in the mentioned square is given by:

$$p_{e-d}=\int_{eL}^{(e+\Delta)L}\int_{dL}^{(d+\Delta)L}f_{X\cos}(x)f_{X\sin}(y)dydx\approx f_{X\cos}(eL)\cdot f_{X\sin}(dL)\cdot(\Delta L)^2=f_{Xv}\left(\sqrt{e^2+d^2}\cdot L\right)\cdot(\Delta L)^2 \quad (19)$$

  - Considering all the elementary squares of the area delimited by the segments considered on the cosine and sine axes, the function which gives the probability that X_V takes the value $\sqrt{e^2+d^2}\cdot L$ (20) may be determined as it is explained in fig. 1.
    - Using an appropriate quantization we can obtain the probability distribution that the value X_V is on arches with radius $\sqrt{2}\cdot\Delta L$ (20)

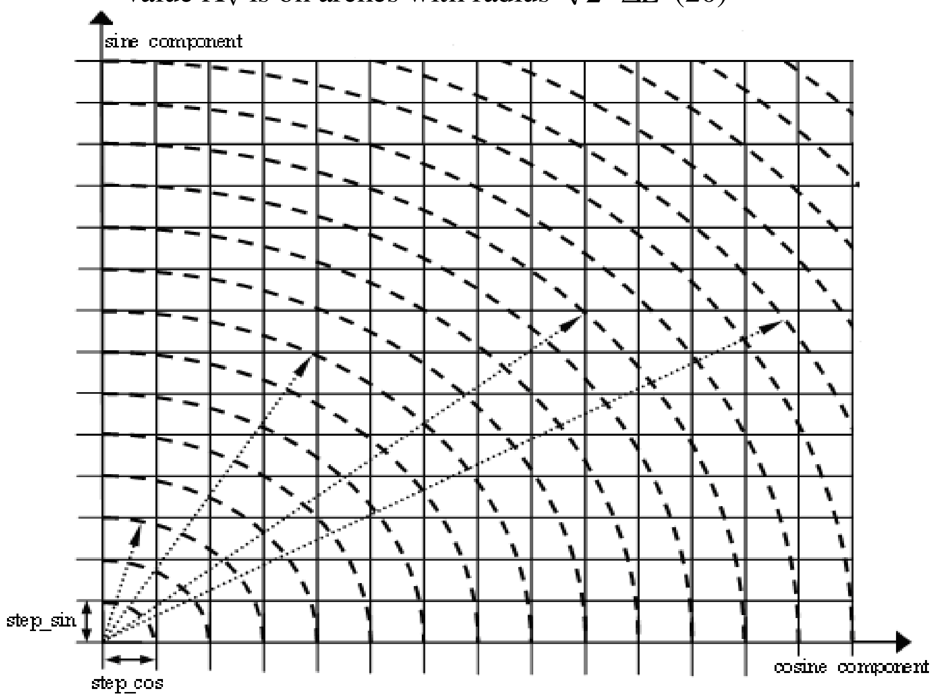


Figure 1 Combination of sine and cosine terms

Multipath propagation models for different frequency bands

2.4GHz –WiFi channel model

RMS Delay Spread	Path 1 Gain (0 dB)	Path 2 Gain (-4 dB)	Path 3 Gain (-8 dB)	Path 4 Gain (-12 dB)	Path 5 Gain (-16 dB)	Path 6 Gain (-20 dB)
0 ns	0.000 ns	0.000	0.000	0.000	0.000	0.000
10 ns	0.000 ns	10.167	20.333	30.500	40.667	50.833
20 ns	0.000 ns	20.500	41.000	61.500	82.000	102.500
30 ns	0.000 ns	30.667	61.333	92.000	122.667	153.333
40 ns	0.000 ns	41.000	82.000	123.000	164.000	205.000
50 ns	0.000 ns	51.167	102.333	153.500	204.667	255.833

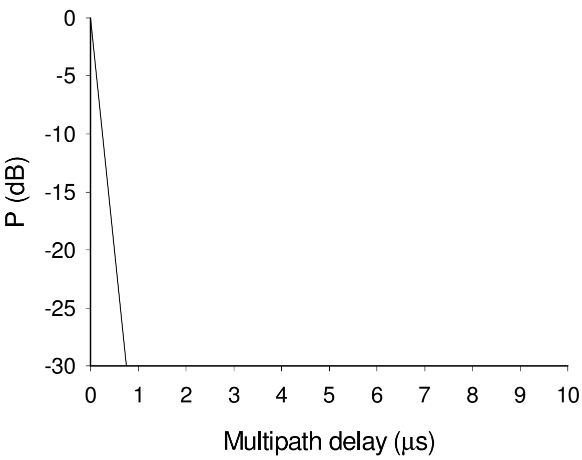
1.9GHz – simple channel model

Path no.	0	1	2	3
Attenuation (dB)	0	3	6	9
Delay (ns)	0	200	400	600

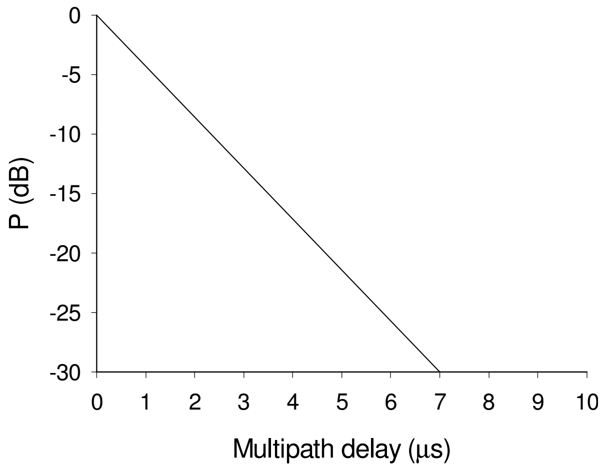
5GHz – simple channel model

	Urban Macro			Urban Micro		
Relative Path Power / Relative Path Power (dB) / Delay (ns)	10 / 20	-3.010	0	7 / 20	-4.559	0
	6 / 20	-5.229	7	5 / 20	-6.021	5
	4 / 20	-6.990	27	4 / 20	-6.990	11
				4 / 20	-6.990	28
Delay-Spread (ns)	10.2			10.0		

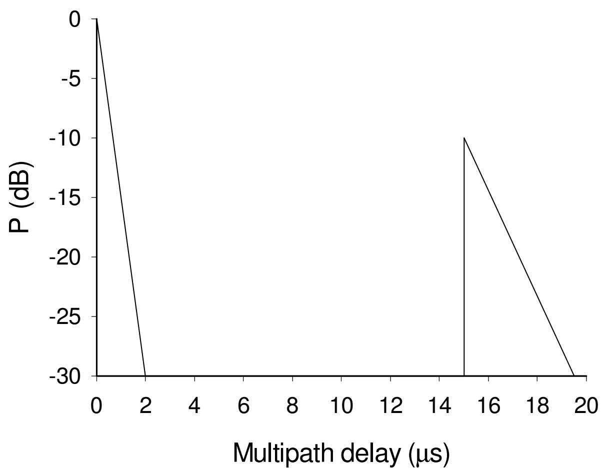
800MHz – 1.8GHz – GSM channel model [IEE]



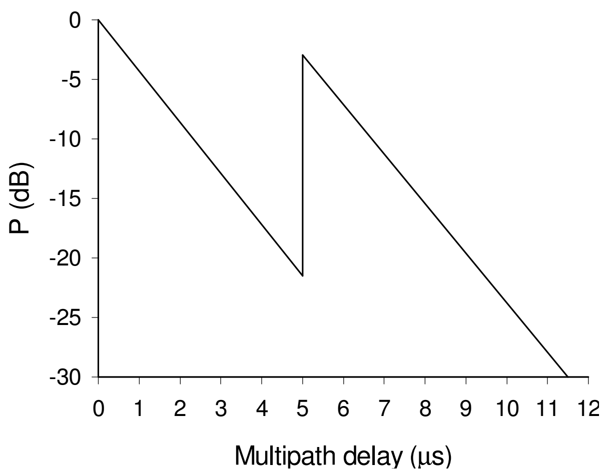
Multipath power delay profile for rural areas



Multipath power delay profile for urban areas



Multipath power delay profile for typical hilly terrain



Multipath power delay profile for bad case hilly terrain

Suburban Macro		Urban Macro		Urban Micro	
Power [dB]	Delay [µs]	Power [dB]	Delay [µs]	Power [dB]	Delay [µs]
-3.0000	0.000	-3.0000	0.000	-4.5500	0.000
-5.2200	0.010	-5.2200	0.010	-6.0000	0.010
-6.9800	0.025	-6.9800	0.030	-6.9800	0.015
-5.6682	0.140	-5.2204	0.360	-6.9800	0.030
-7.8882	0.150	-7.4404	0.370	-5.8161	0.285
-9.6482	0.165	-9.2004	0.385	-7.2661	0.290
-9.2147	0.060	-4.7184	0.250	-8.2461	0.295
-11.4347	0.070	-6.9384	0.260	-8.2461	0.310
-13.1947	0.090	-8.6984	0.280	-7.2701	0.205
-13.4132	0.400	-8.1896	1.040	-8.7201	0.200
-15.6332	0.410	-10.4096	1.045	-9.7001	0.220
-17.3932	0.430	-12.1696	1.065	-9.7001	0.230
-19.4735	1.380	-12.0516	2.730	-8.8473	0.665
-21.6935	1.390	-14.2716	2.740	-10.2973	0.670
-23.4535	1.410	-16.0316	2.760	-11.2773	0.675
-25.1898	2.830	-15.5013	4.600	-11.2773	0.685
-27.4098	2.835	-17.7213	4.610	-10.5640	0.805
-29.1698	2.855	-19.4813	4.625	-12.0140	0.810
				-12.9940	0.820
				-12.9940	0.835
				-12.9806	0.925
				-14.4306	0.935
				-15.4106	0.940
				-15.4106	0.960