Filtering the data signals

Necessity

- one of the effects of band-limitation (by filtering) of a rectangular impulse, of period T_s, is its „extension” in time, which leads to the occurrence of Inter-Symbol Interference (ISI).

- if a_k is an impulse occurring in the k-symbol period, x(t) is the time-response of the filter and τ the delay inserted by the filter, then the signal at the filter's output would be:

$$y(t) = \sum_{i=-n}^{+n} a_{ki} \cdot x(t - kT_s - iT_s - \tau) = a_{k0} x(t - kT_s - \tau) + \sum_{i=-n\ i\neq 0}^{n} a_{ki} \cdot x(t - kT_s - iT_s - \tau); \qquad (1)$$

- i.e., the filtered impulse has a main lobe a_{k0} and a series of side lobes a_{ki}, which occur in the previous symbol periods, i < 0, and in the subsequent symbol periods, i > 0; these side lobes would affect the symbols transmitted during those symbol periods. The amplitudes of the main lobe and of the side lobes depend on the time-response function of the filter employed.

- the signal obtained by filtering a group of data pulses by an LPF filter is presented in figure 1.

- the filtered signal presents significant amount of Inter-Symbol Interference (ISI)

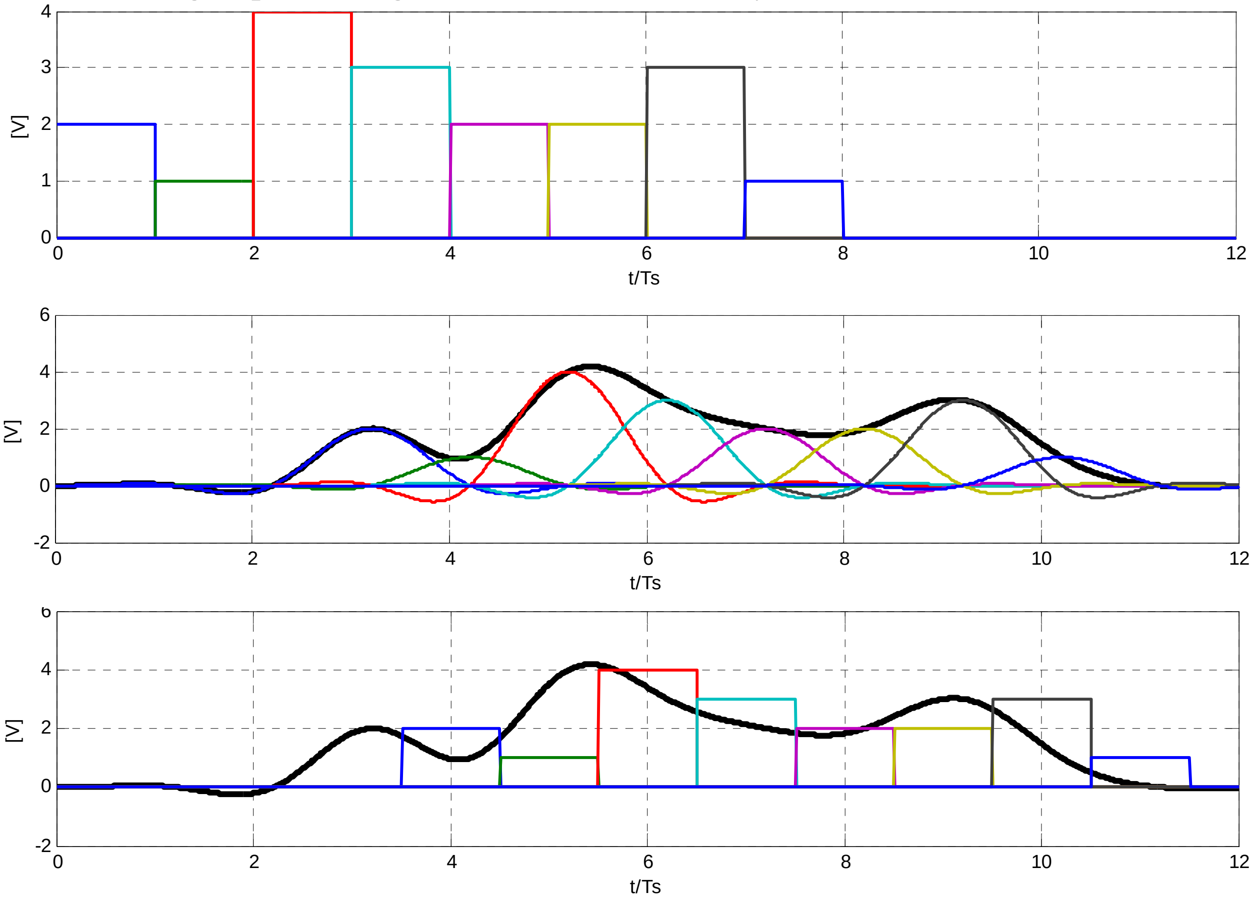

Figure 1 Filtering of data signals with LPF filter that does not ensure ISI cancellation.
a. data levels. b. individual filtered pulses and the resulted signal. c. sampling of the filtered signal

Nyquist filtering criteria

- to avoid the distorting effects of ISI upon the filtered signal, the filter's impulse response should equal zero at well-defined time instants, called probing moments, excepting one, called main probing moment.

- Nyquist showed that in order to transmit symbols with a period Ts, in a frequency band [0, f_N= $1/2T_s$= $f_s/2$] with ISI = 0 in the probing moments, the impulses should be filtered with a filter that has the frequency characteristic and time response defined by relations (2.a) and (2.b), respectively.

- the frequency characteristic and the time-response of (2) are shown in figures 2 and 3, respectively.

$$X(\omega) = \begin{cases} 1; \omega \leq \omega_N; \\ 0; \omega > \omega_N; \end{cases}; \text{ a.} \quad x(t) = \frac{\sin \pi t/T_s}{\pi t/T_s}; \text{ b.} \qquad (2)$$

- this characteristic is called the ideal Nyquist characteristic, because it is not feasible

- the time response of the ideal Nyquist filter equals zero in every symbol period, at the middle of

the symbol period (probing moments), excepting one symbol period, within which at the sampling moment the filtered impulse reaches its nominal value.
- due to this property, in the probing moments the filtered impulse would not affect the values of the impulses transmitted in the previous and subsequent symbol periods, thus ensuring a null ISI in these time instants.

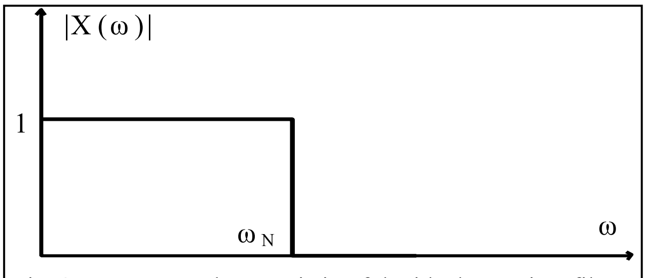

Fig. 2. Frequency characteristic of the ideal Nyquist filter

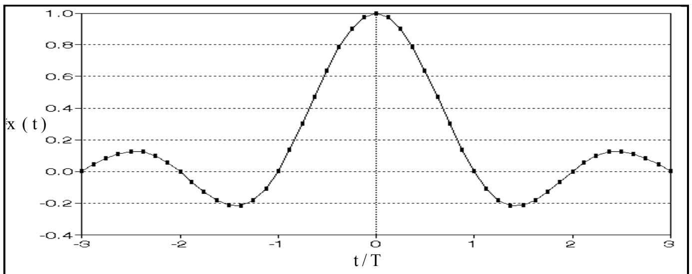

Figure 3. Time response of the ideal Nyquist filter

- to obtain a feasible filtering characteristic, one of the conditions imposed in the Nyquist ideal filter, should be "relaxed"; the three possibilities of doing that are:

a) increasing the BW of the Nyquist filtering characteristic;
b) accepting a controlled non-zero ISI;
c) decreasing the symbol rate;

- fulfillment of condition a) leads to the first Nyquist filtering criterion.
- fulfillment of condition b) leads to the second Nyquist filtering criterion – it generates the so called "Partial Response Techniques".
- fulfillment of condition c) is not be considered since it leads to a) and decreases the bit rate.

Nyquist first filtering criterion. The Raised-Cosine filtering characteristic (RC)
- the frequency characteristic of the this filter is given by (3), where α denotes the "roll-off factor";
- it is the ratio between the additional frequency BW and the minimum required BW (f_N).
- the modulus of the characteristic is represented in figure 4 for α = 0 (the ideal characteristic – approximate), 0.5 and 1.
- because its expression is a squared cosine, this characteristic is named "raised cosine" (RC).
- the ideal characteristic X(ω) (2), which does not require an excess (additional) BW is obtained by making α → 0 in (3).

$$X_\alpha(\omega)=\begin{cases}1; \quad 0\le\omega\le\omega_N(1-\alpha);\\ \dfrac{1-\sin[T_s(\omega-\omega_N)/2\alpha)}{2}=\\ =\cos^2(\dfrac{\pi\omega}{4\alpha\omega_N}-\dfrac{\pi(1-\alpha)}{4\alpha}); \quad \omega\in[\omega_N(1-\alpha),\omega_N(1+\alpha)];\\ 0; \quad \omega>\omega_N(1+\alpha)\end{cases} \tag{3}$$

- expression (3) is a Low-Pass characteristic
- the expression of a RC Band-Pass characteristic, centered on an f_c carrier signal, can be obtained by replacing in (3) ω by (ω – ω_c).
- the frequency BW of the BP-filtered signal is: $B=[\omega_c-\omega_N(1+\alpha);\,\omega_c+\omega_N(1+\alpha)]$**;** (4)
- the impulse response of the RC-filter is defined by (5); it represented in figure 5, for α = 0.5

$$x_\alpha(t)=\frac{\sin\pi t/T_s}{\pi t/T_s}\cdot\frac{\cos\alpha\pi t/T_s}{1-4\alpha^2t^2/T_s^2}; \tag{5}$$

- comparing expressions (2.b) and (5) or figures 3 and 5, we see that the side lobes of the response of the filter with extended bandwidth are significantly smaller than the ones of the ideal filter's response; this is due to the second factor of (5), which is generated by the additional frequency bandwidth employed.
the attenuation of the side lobes increases with the increase of the roll-off factor.

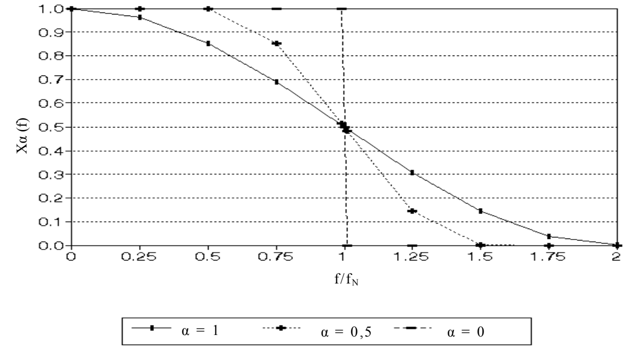

Figure 4. Frequency characteristics of RC filters for α = 1, 0.5, 0

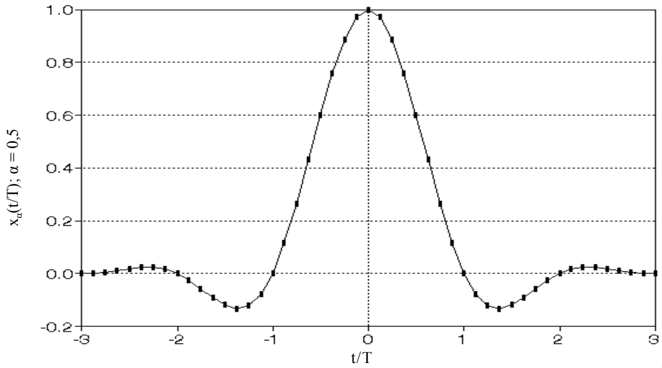

Figure 5. Impulse-response of the RC filter with α = 0.5

- **if $t = kT_s - T_s/2$ is the beginning of the symbol period, then the probing moment is shifted with $T_s/2$ and occurs at the middle of the symbol period; so the probing instants are $t = kT_s$.**
- **the probing moments have the same properties as the one described for the ideal Nyquist filter.**
- **figure 6 presents the filtering of successive data pulses performed by an LPF RC filter.**
- **note that the ISI = 0 in the probing moments, when the filtered signal reaches its nominal values.**

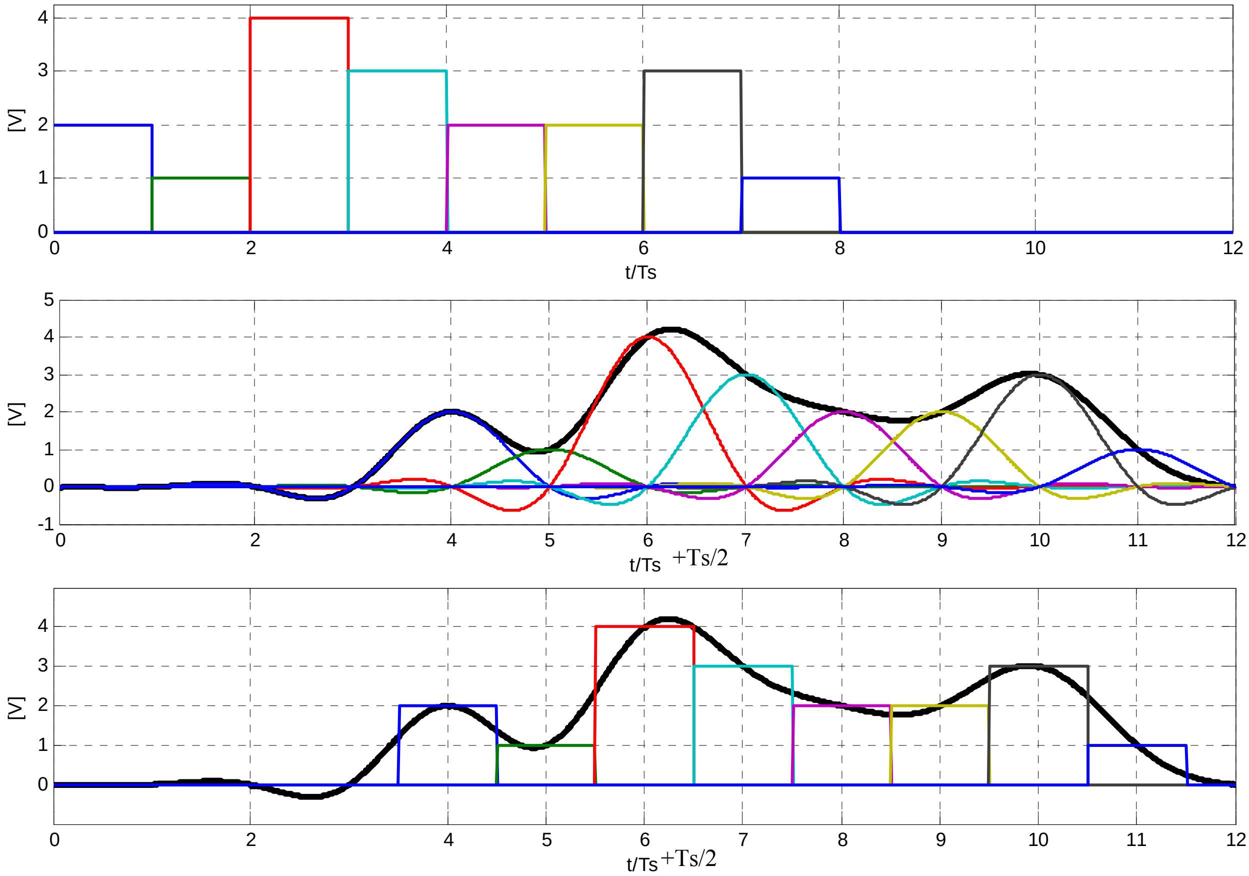

Figure 6 . LPF-RC filtering of data signals.
a. data levels. b. individual filtered pulses and the resulted signal. c. sampling of the filtered signal

The Root-Raised Cosine filtering characteristic (RRC)

- to ensure best performances in the presence of noise, the RC filtering characteristic is equally split between the transmitter and receiver.

- this involves the signal filtering, both at the transmission and receiver ends, with characteristics, G_E and G_R, which equal $X_\alpha^{1/2}$, see (6).

$$X_\alpha(\omega) = G_E(\omega) \cdot G_R(\omega); G_E(\omega) = G_R(\omega) = X_\alpha^{1/2}(\omega) \; ; \qquad (6)$$

- **if the receive filter is placed before the demodulator, at its input the signal is filtered with the product $G_E \cdot G_R$, i.e. an RC characteristic.**
- **the implementation of an RC characteristic is equivalent to the implementation of two RRC characteristics, be them LP or BP-type.**

- **the mathematical expression of the RRC characteristic is expressed by (7) and depicted in figure 7, for α = 0.5, and 1; the ideal Nyquist characteristic is also presented, for reference.**

$$X_\alpha^{1/2}(\omega) = \begin{cases} 1; & 0 \le \omega \le \omega_N(1-\alpha); \\ \cos(\frac{\pi\omega}{4\alpha\omega_N} - \frac{\pi(1-\alpha)}{4\alpha}); & \omega \in [\omega_N(1-\alpha), \omega_N(1+\alpha)]; \\ 0; & \omega > \omega_N(1+\alpha) \end{cases} \quad (7)$$

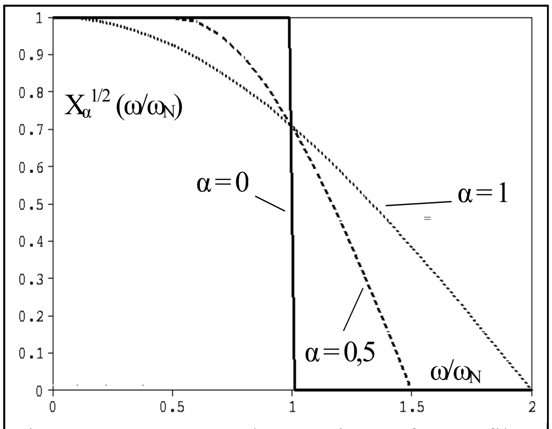

Figure 7. Frequency characteristcs of RRC filters for α = 0.5 and 1 and of the ideal Nyquist filter

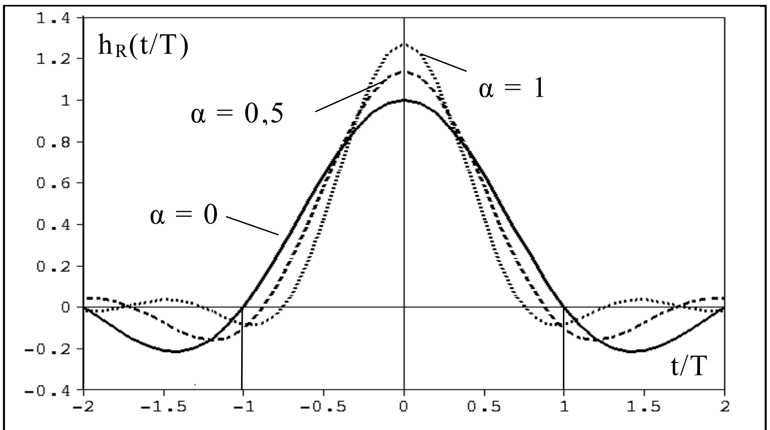

Figure 8. Impulse responses of the RRC filters α = 0.5, 1 and of the ideal al Nyquist filter α = 0

- this characteristic is also named a "cosine" characteristic.
- the characteristic defined by (7) s a LP one; - the BP_RRC can be obtained similarly to the RC characteristic, and the BW of the filtered signal is also given by (4).
- the impulse response of this characteristic is defined by (8) and is shown in figure 8.

$$h_{R\alpha}(t) = \frac{1}{1-(\frac{4\alpha t}{T})^2}[\frac{\sin(\pi(1-\alpha)t/T)}{\pi t/T} + \frac{4\alpha}{\pi}\cdot\cos(\pi(1+\alpha)t/T)]; \quad (8)$$

Note that:

- **the impulse response of the RRC filter, α > 0, does not have null values at the probing instants;**
- **the amplitude of the filtered signal is higher than 1 in the in the main probing moment; the amplitude of the filtered signal in this moment increases with the increase of α.**

- the amplitudes of the side lobes decreases with the increase of α.
- though the filtered signal transmitted in the channel has a non-zero ISI in the probing instants, the signal at the demodulator's input has a ISI = 0, due to the RRC filtering performed in the receiver, i.e. the global impulse response is (4), exhibiting SI =0 n all probing instants, except the main one, t = 0.

Considerations regarding the implementation of the filtering characteristics

- the Nyquist-type characteristics (RC and RRC) can be implemented using either analog or digital filtering structures.
- the analogue implementation with passive components requires the high-order LC filters;
- the design of these filters is complicated due to the requirements imposed to the group-delay time characteristic (ISI=0). The implementation filters require a rather complicated technology.
- the analog implementation using active RC structures involves a rather high number of low-tolerance passive components. Therefore, the analog implementation should be used only for high frequencies, where the digital implementation is not available.
- still, for low roll-off factors, the analog implementation becomes very difficult and inserts relatively figh approximation errors.
- a more adequate method is the digital implementation using Finite Impulse Response (FIR) digital structures that exhibit a linear variation of the phase, in terms of frequency. They can be implemented on Digital Signal Processors or on specialized circuits, due to the relatively high number of taps required.

- This paragraph represents only a simple presentation and does not cover “deeply” the topic.

General aspects

- the FIR transversal structure, shown in figure 9, is described by the finite-differences equation (9), where:

- $x(n-i)$ denotes the current sample of the input signal, sampled with frequency f_e at time $t = nT_e$ and delayed with i sampling periods
- h_i, denote the filter coefficients
- $y(n)$ denotes the current sample of the output filtered signal.

$$y(n) = \sum_{i=0}^{N-1} h_i \cdot x(n-i) \qquad (9)$$

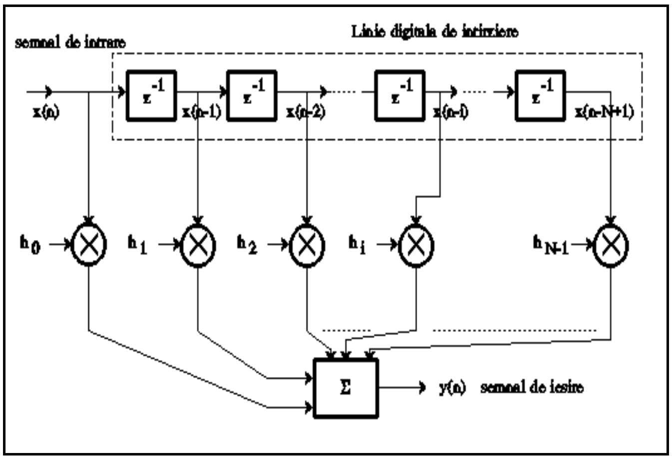

Figure 9. Basic diagram of a FIR transversal filtering structure

- the impulse response of this structure has a finite duration, being null after N sampling periods
- the values of the coefficients represent the samples ($t = iT_e$) of the impulse-response of characteristic to be implemented; this is proven by inputting in (9), unity-impulse of period T_e.
- this structure is stable, because it has no poles in the transfer function

Conditions for the linear phase variation vs. frequency

- it is required to ensure a constant group-delay time vs. frequency characteristic

- if condition (10) is imposed, then a symmetrical impulse-response is obtained, besides the constant group-delay time given by (11):

$$h_i = h_{N-1-i}; \quad (10); \qquad \Phi(\theta) = -\frac{N-1}{2}\cdot\omega T_e; \text{ a. } \tau_g = -\frac{d\Phi}{d\omega} = \frac{N-1}{2}\cdot T_e; \text{ b.} \qquad (11)$$

- if condition (12) is imposed, then an anti-symmetrical impulse is obtained; the phase and group-delay time vs. frequency are given by (13)

$$h_i = -h_{N-1-i}; \quad (12); \qquad \Phi(\theta) = \pm\frac{\pi}{2} - \frac{N-1}{2}\cdot\omega T_e; \text{ a. } \tau_g = \frac{N-1}{2}\cdot T_e; \text{ b.} \qquad (13)$$

Implementation of the RC and RRC filtering characteristics

- the coefficients of the FIR filter are obtained by computing the values of the impulse-response at time instants $t = i\cdot T_e$
- because the impulse responses of the RC and RRC filtering characteristics are spread infinitely in time domain, their time-length is truncated to a finite number L of period symbols.
- because their time-responses are symmetrical, with respect to the main probing moment, $t = 0$, the number of symbol periods L should be even, and $h_i = h_{-i}$, (10), so the group-delay time is constant vs. frequency.
- if we employ m samples per symbol period, and the sampling frequency observes (14), the number of samples (filter’s order) is expressed by (15).

$$f_e = m\cdot f_s \rightarrow T = m\cdot T_e; \; f_s = 1/T; \quad (14); \qquad N = m\cdot L + 1 \qquad (15)$$

- *the RC characteristic*

- by sampling the RC impulse-response (5) at equidistant time-instants $t/T_s = -(mL/2) + i/m$ we get the coefficients h_i (16). Because the symmetry of the impulse-response (10), we need to compute only h_i, for $i \geq 0$

$$h_{\alpha,i} = \frac{1}{m}\cdot\frac{\sin\frac{\pi i}{m}}{\frac{\pi i}{m}}\cdot\frac{\cos\frac{\pi\alpha i}{m}}{1-\left(\frac{2\alpha i}{m}\right)^2}; \; i \in \{0,\ldots,(N-1)/2\}; \; h_{\alpha,-i} = h_{\alpha,i}$$
$$h_{\alpha,0} = \frac{1}{m}; \; h_{\alpha,m/2\alpha} = \frac{1}{2m}\sin\frac{\pi}{2\alpha}; \quad \text{for } \frac{m}{2\alpha} \in \mathbb{N} \qquad (16)$$

- the coefficients with indexes i = 0 and m/2α, for m/(2α)∈N, should be computed using the l'Hospital rule, see (16)
- the group-delay time inserted by this structure is:

$$\tau_g = T_e(N-1)/2 = T_s(N-1)/(2m) = T_s \cdot L/2; \qquad (17)$$

-the RRC characteristic

– by sampling the time-response of the RRC characteristic (8), we get the coefficients of the FIR structure that implements it, (18). The number of coefficients are computed similarly as for the RC.

$$h_{R\alpha i} = \frac{1}{m} \cdot \frac{1}{1-\left(\frac{4\alpha i}{m}\right)^2} \cdot \left\{ \frac{\sin\left[\frac{\pi i}{m}(1-\alpha)\right]}{\frac{\pi i}{m}} + \frac{4\alpha}{\pi}\cos\left[\frac{\pi i}{m}(1+\alpha)\right] \right\}; \; i \in \{0,.., \frac{N-1}{2}\}$$
$$h_{R\alpha 0} = \frac{1}{m}\left(1-\alpha+\frac{4\alpha}{\pi}\right); \qquad (18)$$

- the coefficients with indexes i = 0 and i = m/(4α), for m/(4α)∈N, should be computed using the l'Hospital rule, see (18)
- the delay inserted by this structure is also given by (17).