# Pulse Amplitude Modulation

- **Symbol frequency** – in the digital transmission the symbol frequency (or the signaling frequency) equals the number of variations per time unit (second) of the modulated parameter (parameters), produced by the modulation process.
- A symbol from the signaling alphabet could „carry” $n$ bits
- The symbol period may be defined as the minimum time interval during which the value of the modulated parameter remains constant. This assertion holds for the amplitude shift keying (*ASK*) and frequency shift keying (*FSK*). For the phase shift keying (*PSK*), the modulated parameter has a linear variation during a symbol period.
- Considering the bit frequency at the modulator's inputs is $f_{bit}$, the symbol period $T_s$ is expressed by (1), while the symbol frequency $f_s$ is expressed by (2).

$$T_S = n \cdot T_{bit} = \frac{n}{f_{bit}} \qquad (1)$$

$$f_s = \frac{1}{T_s} = \frac{f_{bit}}{n} \qquad (2)$$

- If $n$ bits are mapped (loaded) on a symbol, the modulated parameter can have one of the $M=2^n$ possible distinct values, which compose the modulation (channel) alphabet $\mathscr{M}$.
- The mapping operation performs the correspondence between the $M$ combinations of $n$ bits and the $M$ values of the modulated parameter, i.e. the modulation alphabet $\mathscr{M}$, according to a *mapping rule*
- The receiver performs the *demapping* function, according to the *demapping rule*, providing the binary $n$ bit-long sequences corresponding to the received modulating symbol.

***Pulse Amplitude Modulation (PAM)***

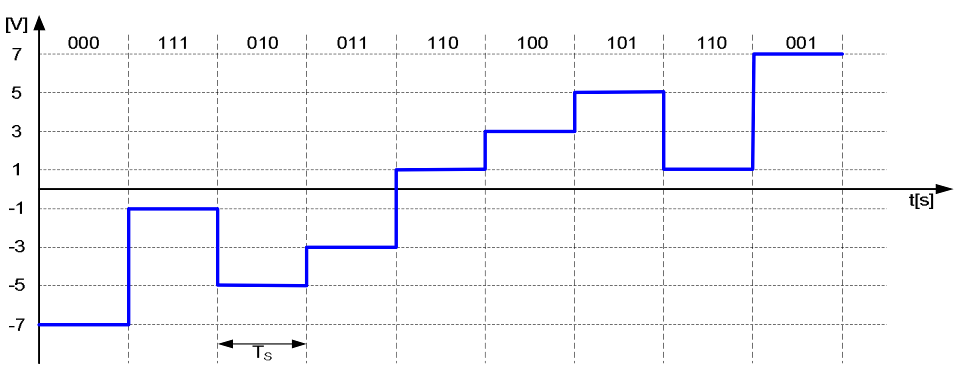
- The digital *PAM* transmits during the *k-th* symbol period a constant amplitude $m_k$, which could be one of the $M=2^n$ values belonging to the modulation alphabet M .
- Imposing that the average value of the modulated signal equals zero, condition that leads to symmetrical levels, and that the difference between two neighbouring levels to be constant, the possible levels are defined by:

$$m_i = (2i+1-M)\cdot A_0; \quad i = 0,1,\ldots,2^n-1 \qquad (3)$$

where $A_0$ is a scaling constant and $2A_0$ is the minimum distance between any elements of M
- The maximum value of the modulated level is:

$$|A_{\max}| = (M-1)\cdot A_0 = (2^n-1)\cdot A_0 \qquad (4)$$

- Figure 1 presents an example of PAM signal and the mapping function (table) employed for $n$ = 3 and $A_0$=1V. The tri-bit to level mapping is mapping according to the Gray rule, so that the tribits corresponding to adjacent levels differ by only one bit.




| bitii | $\mathscr{M}$[V] |
|---|---|
| 000 | -7 |
| 010 | -5 |
| 011 | -3 |
| 111 | -1 |
| 110 | 1 |
| 100 | 3 |
| 101 | 5 |
| 001 | 7 |

Figure 1. Example of PAM and mapping table

- If we assume that the levels of the modulated signal (3) are equiprobable, the average power of the *PAM* signal (base band) is expressed by (5). This relation is obtained using the sum of the first $M$ natural numbers and the sum of their squares:

$$P_{mPAM} = \frac{A_0^2 \cdot \sum_{i=0}^{M-1}(2i+1-M)^2}{M} = \frac{A_0^2\cdot(M^2-1)}{3}; \text{Coment on } A_0 \qquad (5)$$

- The expression of the *PAM* signal is:

$$s_{PAM}(t)=\sum_{k=0}^{\infty} m_k \cdot u_T(t-kT_S) \quad (6)$$

where $u_T(t)$ is a unity-step impulse which has duration of one symbol period $T_S$, expressed by (7).

$$u_T(t)=\begin{cases}1; & t\in[0;T_s)\\ 0; & t\notin[0;T_s)\end{cases} \quad (7)$$

- Considering that the modulated signal may take $M(=2^n)$ amplitude levels and the modulating sequences are random, the amplitude of the *PAM* signal is a random variable of mean equaling $m_m$ and variance $\sigma_m$. Due to the finite duration of each level, see (7), the power spectral density of the *PAM* signal is:

$$S_{PAM}(f)=\sigma_m^2\cdot T\cdot\left(\frac{\sin\frac{\pi f}{f_s}}{\frac{\pi f}{f_s}}\right)^2+m_m^2\cdot T\cdot\sum_{k=-\infty}^{\infty}\left[\left(\frac{\sin\frac{\pi(f-kf_s)}{f_s}}{\frac{\pi(f-kf_s)}{f_s}}\right)^2\cdot\delta(f-kf_s)\right] \quad (8)$$

- Due to the zero-mean condition imposed, i.e., $m_m=0$, which makes the modulated levels to be symetrical w.r.t. 0V, the spectral power density of the *PAM* signals is expressed only by the first term of (8); actually, $\sigma_m^2$ is proportional to the average power given by (5)
- The amplitude spectrum of the *PAM* signal is represented in Figure 2

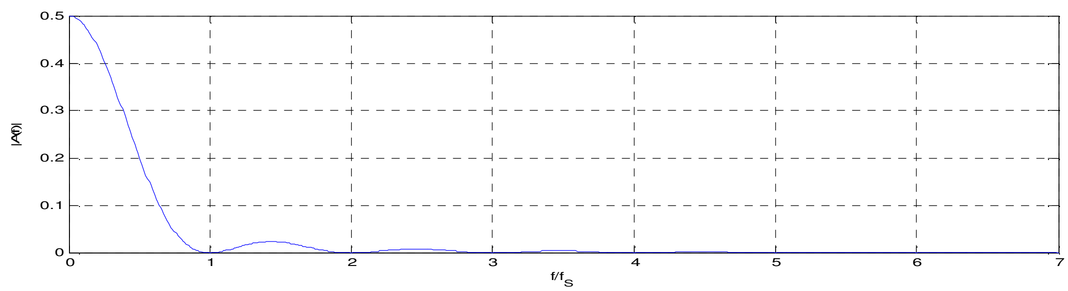


Figure 2 Spectral amplitude density of the PAM signal

***Optimal Demodulation of the PAM signals on AWGN channels***

- The signal received during a symbol period on a *AWGN* channel can be expressed as:

$$r_k(t)=m_k\cdot u_{T_s}(t-kT_s)+n(t);\quad m_k\in\mathscr{M} \quad (9)$$

- Since the modulating signal $m_k$ is constant over a symbol period, the received signal should be averaged over a that symbol period, as shown in Figure 3.

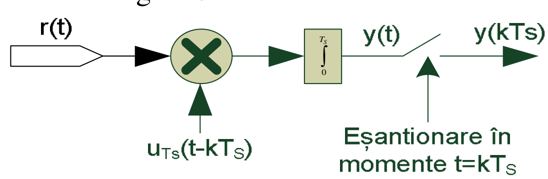


Figure 3. Correlator-based PAM demodulator

- The correlation between the received signal and the step-impulse, which acts like a „time-window" that marks-off the integration intervals of each symbol by reseting the integrator at the beginning of each symbol period, is expressed by:

$$\begin{aligned}y(t)&=\int_0^t r(\tau)\cdot u_{T_S}(\tau-kT_S)\,d\tau=\int_0^t\left[m_k\cdot u_{T_s}(\tau-kT_s)+n(\tau)\right]\cdot u_{T_S}(\tau-kT_S)\,d\tau=\\&=m_k\int_0^t\left[u_{T_s}(\tau-kT_s)\right]^2d\tau+\int_0^t n(\tau)\cdot u_{T_s}(\tau-kT_s)\,d\tau\end{aligned} \quad (10)$$

- By sampling the integrator's output at the end of the symbol period we get:

$$y(kT_S)=m_k+n_k \quad (11)$$

- the „noise" term $n_k$ has a Gaussian distribution:

$$n_k=\int_0^{T_S} n(\tau)\cdot u_{T_s}(\tau-kT_s)\,d\tau \quad (12)$$

- Assuming that the noise power spectral density equals $N_0$, the variance of the noise samples $n_k$ would be

(13); this shows that the noise at the demodulator's output still has a constant power spectral distribution and its amplitude's distribution is Gaussian

$$\sigma^2 = P_n = \int_0^{T_S} (n_k)^2\, dt = \int_0^{T_S} \left( \int_0^{T_S} n(\tau)\cdot u_{T_s}(\tau - kT_s)\, d\tau \right)^2 dt = \\ = N_0 \int_0^{T_S} \left( u_{T_s}(t-kT_s) \right)^2 dt = N_0 \qquad (13)$$

***Symbol error probability of PAM***

- Assuming that the received *PAM* signal is affected by a Gaussian Additive White Noise (*AWGN*) of zero-mean and variance $\sigma$, its expression becomes:

$$r_{PAM}(t) = s_{PAM}(t) + n(t) \qquad (14)$$

- The conditional probability that the received would equal *r* in the sampling moment, if the transmitted level is $m_i$, is:

$$p(r \mid m_i) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \exp\left( -\frac{(r-m_i)^2}{2\sigma^2} \right); \qquad (15)$$

- The distributions of the probability densities of the *M* transmitted levels that are affected by the *AWGN* channel, and the values of the *M-1* decision thresholds are presented in Figure 4.

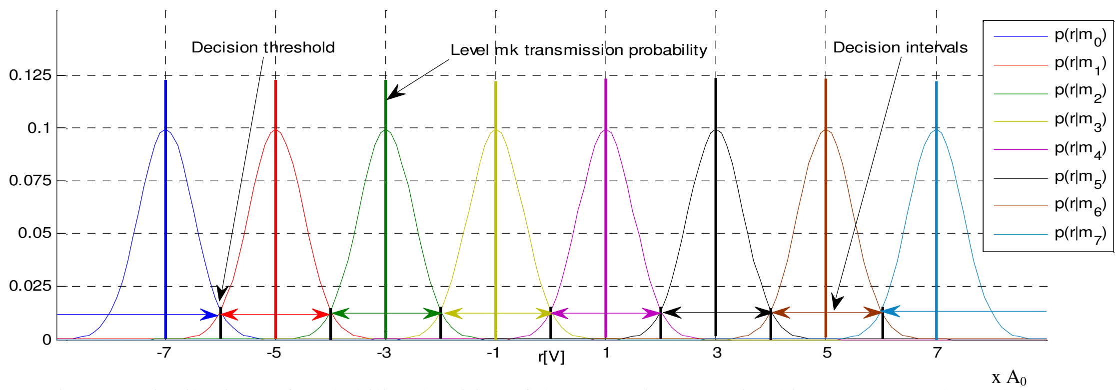



Figure 4 Distributions of probability densities of the PAM signal received in an AWGN channel; decision thresholds and intervals

- Since the decided symbols are obtained by selecting the permitted symbol that has the minimum (Euclidean) distance to the received level, the symbol error probability $p_e$ equals the occurrence probability of a noise level that would make the level of the received signal, i.e. transmitted signal +noise signal, to get closer to another permitted level than to the transmitted one.

- If during *k-th* symbol period the transmitted level is $m_k$ and its occurrence probability is $P_{mk}$, then the symbol-error probability is:

$$p_e = \sum_{k=1}^{M} \left( P_{mk} \cdot p\left( |r-m_k| > A_0 \right) \cdot N_{k,A_0} \right); \qquad (16)$$

where $N_{k,A_0}$ denotes the number of permitted levels placed at $d_E = 2A_0$ from level $m_k$, i.e. $N_{k,A0} = 1$ or 2.

- Note that if the modulating data are randomized, then the transmitted levels have equal occurrence probabilities, i.e. $P_{mk} = \frac{1}{M}$

- Assuming now that the transmitted levels are according to equation (3), then (16) may be rewritten as:

$$p_e = \frac{1}{M}\cdot\left[ p\left(|r-m_1| > A_0\right) + p\left(|r-m_M| > A_0\right) + \sum_{k=2}^{M-1} \left( p\left(|r-m_k| > A_0\right)\cdot 2 \right) \right] = \\ = \frac{2(M-2)+2}{M} p\left(|r-m_k| > A_0\right) = \frac{2(M-1)}{M} p\left(|r-m_k| > A_0\right); \qquad (17)$$

- Inserting now (15) in (17), the average symbol error probability becomes:

$$p_e = \frac{2(M-1)}{M}\cdot\frac{1}{\sqrt{2\pi}\cdot\sigma} \int_{A_0\cdot\sqrt{2}}^{\infty} \exp\left( -\frac{(r-m_k)^2}{2\sigma^2} \right) d(r-m_k) = \frac{2(M-1)}{M}\cdot Q\left(\frac{\sqrt{2}\cdot A_0}{\sigma}\right); \qquad (18)$$

where:

$$Q(t)=\frac{1}{\sqrt{2\pi}}\int_t^{\infty}\exp\left(-\frac{u^2}{2}\right)du;$$

- The symbol-error probability can also be expressed in terms of the Signal-to-Noise Ratio (SNR) in its linear representation, $\rho$, by using (5):

$$p_e=\frac{2(M-1)}{M}Q\left(\sqrt{\frac{6P_m}{(M^2-1)\sigma^2}}\right);\quad \rho=\frac{P_m}{\sigma^2} \tag{19}$$

- The *Q(t)* function could be approximated, by using its Taylor series expansion, as:

$$\mathrm{Q(t)}=\frac{1}{2}\mathrm{erfc}\left(\frac{t}{\sqrt{2}}\right)=\frac{1}{\sqrt{2\pi}}\int_t^{\infty}e^{-\frac{u^2}{2}}du\approx\frac{e^{-\frac{t^2}{2}}}{t\sqrt{2\pi}}\left(1-\frac{1}{t^2}+\frac{3}{t^4}-\ldots\right); \tag{20}$$

- For medium and great values of the argument, the *Q* function can be approximated by the first term of its Taylor-series expansion (20) and so the symbol error probability could be approximated by:

$$p_e\approx\frac{2(M-1)}{M}\cdot\frac{\sqrt{M^2-1}}{\sqrt{6}}\cdot\frac{e^{-\frac{6}{M^2-1}\cdot\frac{\rho}{2}}}{\sqrt{2\pi\rho}}; \tag{21}$$

- The bit-error probablity $p_b$ or *BER* also depends on the multibit-level mapping rule.
- Expression (18) shows that the probability of a given level $m_k$ to be mistaken for one of its neighbouring levels is greater than than its probability to be mistaken for one of the levels placed at greater distances than $2A_0$.
- Then, if the multibit-level mapping is made according to Gray rule, the bit-error probability may be rather accurately approximated by (22) at high and medium values of $\rho$; in (22) $n$ denotes the number of bit/symbol, i.e. n = ld M or $\log_2 M$. This approximation assumes that a certain level is mistaken only for its neighbouring levels and hence the wrong decision of that level leads to only one bit in error

$$BER=\frac{N_{be}}{N_{br}}\approx\frac{1\cdot N_{se}}{n\cdot N_{sr}}=\frac{p_e}{n} \tag{22}$$

- The Gray mapping ensures a bit error probability smaller the mapping made according to a natural-binary code.
- This statement is justified by considering the 4-PAM, i.e. the levels $\{+/-3A_0, +/-1A_0\}$.
- Assuming that level +1 is transmitted, its wrong decison to levels +3 or -1 occurs with the same probability $p_{e1}$, and its wrong decision to level -3 occurs with probability $p_{e2}$. Using (18) we get that:

$$d(+1;+3)=d(+1;-1)<d(+1;-3)\Rightarrow p_{e1}>p_{e2} \tag{23}$$

- With a Gray mapping the dibit-level correspondence is ((-3 ↔ *00*; -1 ↔ *01*; +1 ↔ *11*; +3 ↔ *10*) and for mapping acc. to the natural-binary code the correspondence is (-3 ↔ *00*; -1 ↔ *01*; +1 ↔ *10*; +3 ↔ *11*)
- Then using the Gray mapping, the average bit-error probability is:

$$p_{bG}=2\cdot p_1\cdot 1+p_2\cdot 2 \tag{24}$$

- while for the natural-binary mapping the average bit-error probability is:

$$p_{bn}=p_1\cdot 1+p_1\cdot 2+p_2\cdot 1 \tag{25}$$

- The level-error probabilities $p_1$ and $p_2$ have the same values for both mapping rules
- Comparing the bit-error probabilities of the two mapping rules we get:

$$p_{bG}-p_{bn}=p_2-p_1<0\ \Rightarrow p_{bG}<p_{bn} \tag{26}$$

- The transmission of PAM signals in band-limited channels (e.g. radio channels) requires their filtering in order to limit their bandwith. Since the filtering is performed with a particular filtering characteristic called Raised Cosine, which is presented in the Filtering the Data Signals lecture, considerations regarding the transmission of PAM signals in such channels will be presented at the end of that lecture.