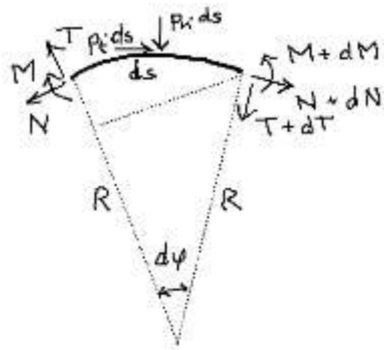
DIFFERENTIAL REALTIONS BETWEEN LOADS AND STRESSES

By any planar curved beam, regardless of the shape of its curvature, a segment with an almost negligible length (ds) may be considered. With such a small length, its curvature can be considered circular in shape (having R radius), while the load on the segment can be considered as a portion of a uniformly distributed force (having the p intensity and a certain orientation). The intensity of this evenly distributed load can be decomposed into a p_t tangential component and a p_n normal component relative to the direction of the axis of the beamr segment. As a result of this load, at the ends of the segment the stress state will be different. In order to determine the difference between the stress states at the two ends of the segment, the conditions of static equilibrium can be expressed (by equations of projections and bending moment) as:



$$N + dN = N \cdot \cos d\varphi + T \cdot \sin d\varphi - p_t \cdot ds \cdot \cos\frac{d\varphi}{2} - p_n \cdot ds \cdot \sin\frac{d\varphi}{2}$$

$$T + dT = T \cdot \cos d\varphi - N \cdot \sin d\varphi + p_t \cdot ds \cdot \sin\frac{d\varphi}{2} - p_n \cdot ds \cdot \cos\frac{d\varphi}{2}$$

$$M + dM = M + T \cdot ds \cdot \cos d\varphi - N \cdot ds \cdot \sin d\varphi + p_t \cdot ds \cdot \frac{ds}{2} \cdot \sin\frac{d\varphi}{2} - p_n \cdot ds \cdot \frac{ds}{2} \cdot \cos\frac{d\varphi}{2}$$

Given that the ds length of the segment and the $d\varphi$ angle are wery small, we can consider $\sin\frac{d\varphi}{2} \cong 0$ and $\frac{ds}{2} \cong 0$, respectively $\cos\frac{d\varphi}{2} = 1$ and $\cos d\varphi \cong 1$. Consequently, neglecting almost zero values, from the above equations will result:

$$dN = -p_t \cdot ds + T \cdot \sin d\varphi$$
$$dT = -p_n \cdot \mathrm{ds} - N \cdot \sin d\varphi$$
$$dM = T \cdot ds$$

Considering $\sin d\varphi \cong d\varphi$ and $d\varphi = \frac{ds}{R}$ we obtain:

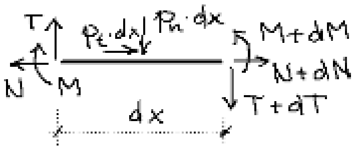
$$dN = -p_t \cdot ds + T \cdot \frac{ds}{R}$$
$$dT = -p_n \cdot ds - N \cdot \frac{ds}{R}$$
$$dM = T \cdot ds$$

Hence the following differential relations:

$$\frac{dN}{ds} = -p_t + \frac{T}{R}$$
$$\frac{dT}{ds} = -p_n - \frac{N}{R}$$
$$\frac{dM}{ds} = T$$

It can be noticed that it was not specified if it is a statically determined or statically indeterminate structure from which the small ds length curved beam segment was considered, so the above relations are valid for all planar structure types.

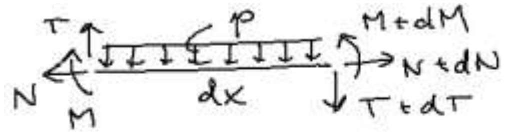
In the case of a straight segment ($R \to \infty$) with a small dx length, the above relations will look like this:



$$\frac{dN}{dx} = -p_t$$
$$\frac{dT}{dx} = -p_n$$
$$\frac{dM}{dx} = T$$

In other words, the variation of the tangent of the axial stress diagram depends on the tangential component of the load, the variation of the tangent of the shear force diagram depends on the normal component of the load, and the variation of the tangent of the bending moment diagram depends on the shear force.

In the case of a normal load on the beam axis, without a tangential component ($p_t = 0$), the previous relations will look as:

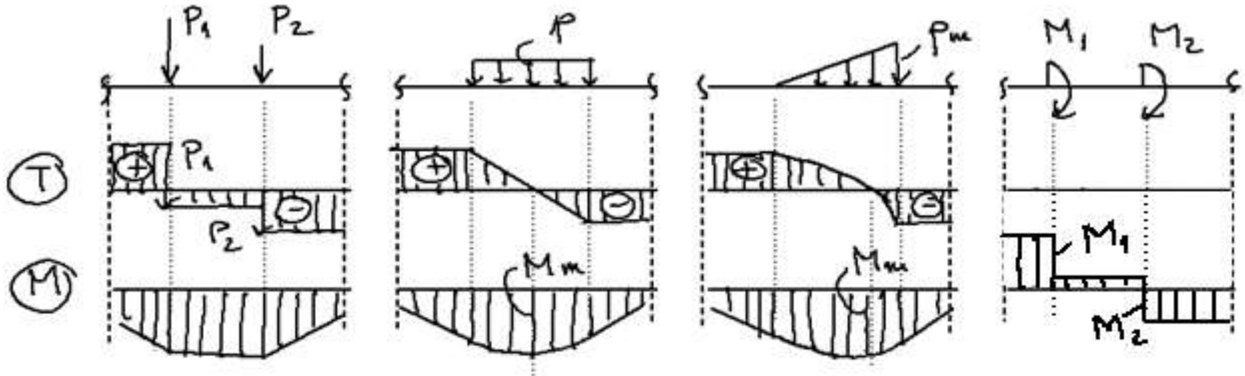


$$\frac{dN}{dx} = 0$$

$$\frac{dT}{dx} = -p$$

$$\frac{dM}{dx} = T$$

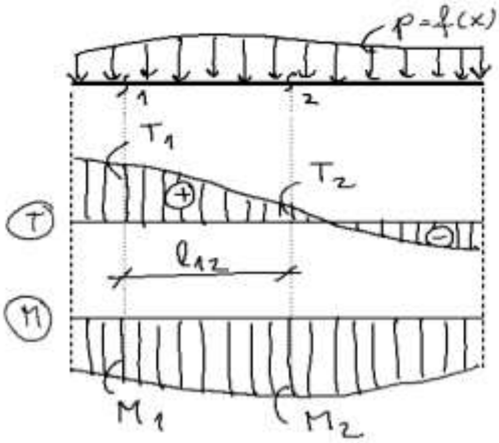
Here are some examples of the differential relationships between loads and stresses, illustrated by shear force and bending moment diagrams:



Under concentrated forces (point loads) perpendicular to the axis of the bar, there is a jump in the shear force diagram and a peak (sudden change of the tangent's direction) in the bending moment diagram. An evenly distributed perpendicular load will lead to a linear variation of the shear force and a parabolic variation of the bending moment diagram (with a horizontal tangent at the point where the shear force passes through the reference line). A distributed perpendicular load with a linearly variable intensity will lead to a parabolic variation (of 2^{nd} degree) in the shear force diagram and to a 3^{rd} degree variation in the bending moment diagram. The load with concentrated bending moments (as point loads) will not affect the shear force diagram, but will cause jumps in the bending moment diagram (the tangent of the bending moment diagram being constant, in accordance with the shape of the shear force diagram).

RECCURENCE RELATIONSHIPS FOR STRESSES

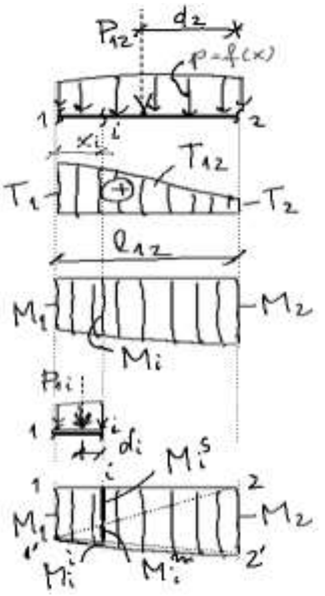
Consider a straight beam segment with a perpendicular load variably distributed on its axis, the load having the intensity $p = f(x)$, as shown in the figure below. Noting that from the differential relations discussed above, in the case of a straight beam with an evenly distributed normal force load the change in values of shear force and bending moment at the ends of a segment can be expressed as $dT = -p \cdot dx$ and $dM = T \cdot dx$, in section 2 these efforts may be expressed in terms of their values in section 1, as follows:



$$T_2 = T_1 - \int_1^2 p \cdot dx \quad \text{și} \quad M_2 = M_1 + \int_1^2 T \cdot dx$$

The integral on the interval 1–2 represents the area of the of the p load diagram, respectively the area of the T diagram between the two sections, which leads us to the expression of the stresses in section 2 as:

$$T_2 = T_1 - P_{12} \quad \text{și} \quad M_2 = M_1 + T_{12}$$



On the other hand, isolating segment 1–2, from the condition of static equilibrium the bending moment in section 2 can also be expressed as:

$$M_2 = M_1 + T_1 \cdot l_{12} - P_{12} \cdot d_2$$

where P_{12} is the resultant of the $p = f(x)$ load on segment 1–2, and d_2 is the distance from section 2 of this resultant. Using this relation to express the value of the shear force in section 1, we obtain:

$$T_1 = \frac{M_2 - M_1}{l_{12}} + P_{12} \cdot \frac{d_2}{l_{12}}$$

Considering another section i on segment 1–2, the value of the bending moment in this point can be written similarly, expressing the static equilibrium:

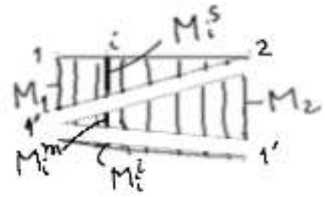
$$M_i = M_1 + T_1 \cdot x_i - P_{1i} \cdot d_i$$

where P_{1i} is the resultant of the $p = f(x)$ load on segment 1–i, and d_i is the distance from section i of this resultant. By replacing T_1 in this expression with the previous relation, we obtain:

$$M_i = M_1 + \left(\frac{M_2 - M_1}{l_{12}} + P_{12} \cdot \frac{d_2}{l_{12}}\right) \cdot x_i - P_{1i} \cdot d_i =$$

$$= M_1 + (M_2 - M_1) \cdot \frac{x_i}{l_{12}} + P_{12} \cdot \frac{d_2}{l_{12}} \cdot x_i - P_{1i} \cdot d_i$$

On the other hand, the expression of the value of this bending moment can also be expressed geometrically (from the surface of the bending moment diagram corresponding to the segment 1-2), dividing the surface into two triangles (1-1'-2, 1-2'-2) and a third surface with a curved face



(below the 1'-2' line). Thus, the bending moment in section i will be:

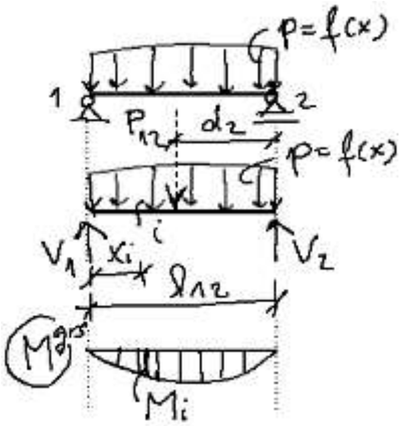
$$M_i = M_i^s + M_i^m + M_i^i = \frac{(l_{12} - x_i)}{l_{12}} \cdot M_1 + \frac{x_i}{l_{12}} \cdot M_2 + M_i^i =$$

$$= M_1 - \frac{x_i}{l_{12}} \cdot M_1 + \frac{x_i}{l_{12}} \cdot M_2 + M_i^i = M_1 + (M_2 - M_1) \cdot \frac{x_i}{l_{12}} + M_i^i$$

Comparing this relationship with the previous one written for M_i, it results:

$$M_i^i = P_{12} \cdot \frac{d_2}{l_{12}} \cdot x_i - P_{1i} \cdot d_i$$

In case of a simple supported beam with l_{12} length, loaded with $p = f(x)$, the bending moment in a section i (located at x_i distance from the left end, marked 1) would result from the condition static



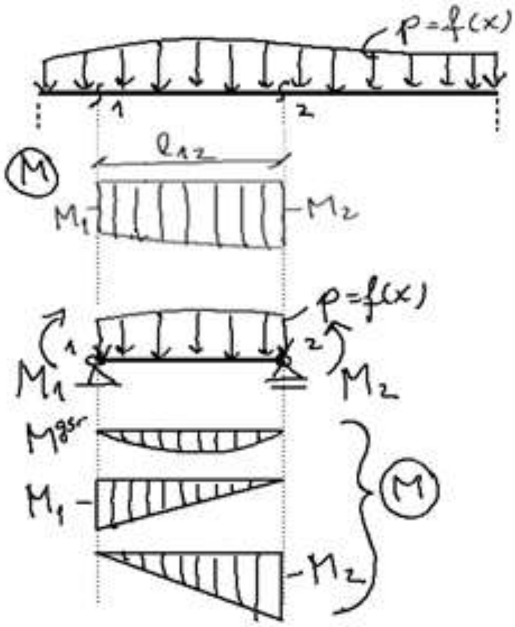
equilibrium as:

$$M_i^{gsr} = V_1 \cdot x_i - P_{1i} \cdot d_i$$

Where the reaction at end 1 would have the expression:

$$V_1 = P_{12} \cdot \frac{d_2}{l_{12}}$$

In conclusion, the lower segment of M_i (marked as M_i^i) is in fact the value of the bending moment in the point corresponding to section i of segment 1–2, located on a simple supported beam with a span equal to l_{12}, loaded with $p = f(x)$ (identically with the loading on segment 1-2). In other words, if two triangles are extracted from a bending moment diagram as in the example shown, the remaining portion coincides with the bending moment diagram from a simply supported beam with the same load.



Consequently, any portion of the straight beam can be extracted from a structure (with the loads corresponding to the portion) and treated as a simple supported beam which, in addition to the related loads, will also be actuated by the M_1 and M_2 bending moments at its ends.