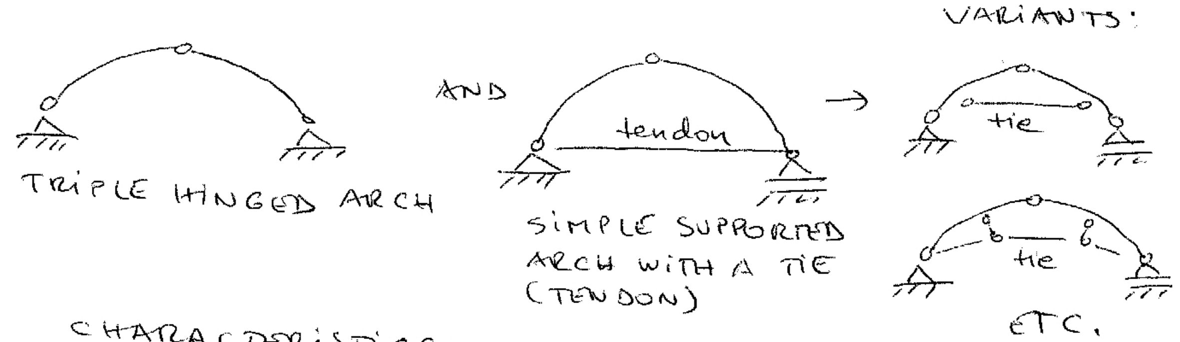
LECTURE 4

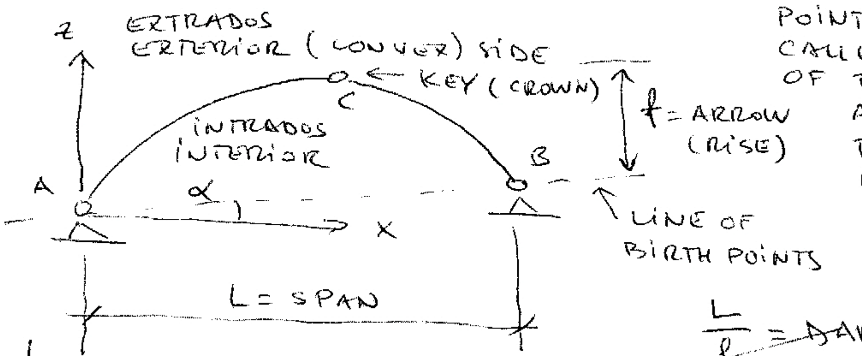
ARCHES

PLANE ARCHES ARE CURVED STRUCTURES LOADED IN THEIR PLANE (USUALLY FROM THE CONVEX SIDE). THEY ARE WORKING THROUGH AXIAL EFFORTS (COMPRESSION), NOT LIKE CURVED BEAMS WICH ARE WORKING THROUGH BENDING MOMENT. DUE TO THE INTERIOR COMPRESSION, AT THE END SUPPORTS OF PLANE ARCHES THERE WILL BE LATERAL PUSHING FORCES.

STATICALLY DETERMINED PLANE ARCHES:



CHARACTERISTICS:



POINTS A AND B ARE CALLED BIRTH POINTS OF THE ARCH (THE AB LINE IS CALLED THE LINE OF BIRTH POINTS), ALSO ABUTTEMENTS. THE HEIGHT IS CALLED ARROW (f), OR THE RISE OF THE ARCH. THE HIGHEST POINT IS THE CROWN (OR KEY).

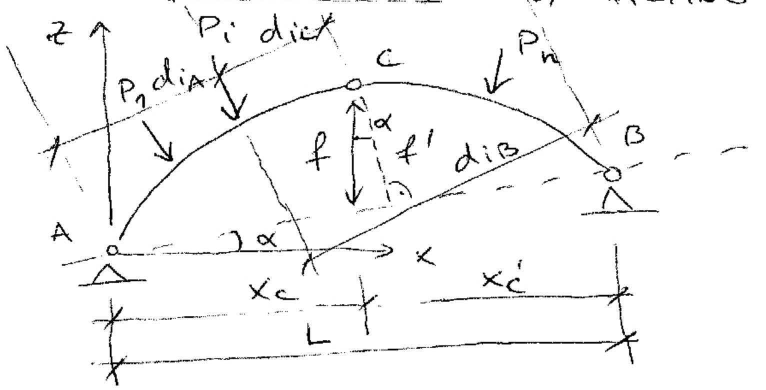
$\frac{L}{f}$ ~~= DAR~~

$\frac{L}{f}$ = BOLDNESS COEFFICIENT (SPAN-RISE RATIO)

$\frac{f}{L}$ = FLATTNESS (RISE-SPAN RATIO)
- → FLAT ARCHES: $\frac{f}{L} \leq \frac{1}{5}$
- → HIGH RISED ARCHES: $\frac{f}{L} > \frac{1}{5}$

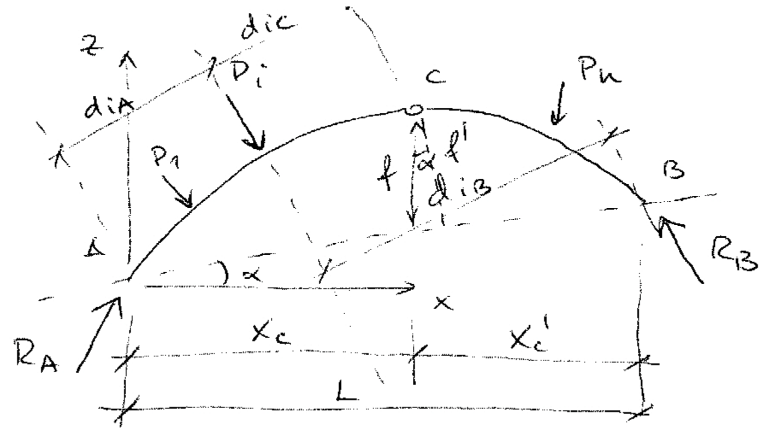
SOLVING METHODS FOR TRIPLE HINGED PLANE ARCHES

1) ANALYTICALLY - BY APPLYING STATIC EQUILIBRIUM EQUATIONS



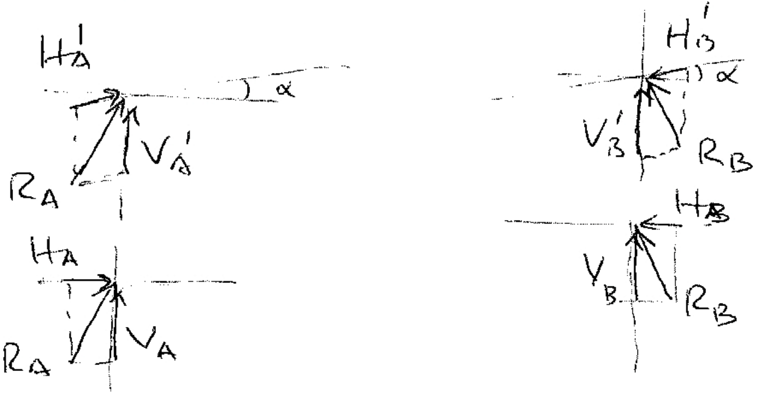
THE STATIC SCHEME SHOULD BE DRAWN FIRST, MARKING ALL THE LOADS ($P_1 \dots P_i \dots P_n$).

THE DISTANCES FROM POINT A (d_{iA}), C (d_{iC}) AND B (d_{iB}) ARE MARKED FOR THE P_i FORCE.



THE SUPPORTS ARE REPLACED WITH THE PROPER REACTIONS (R_A AND R_B).

THUS, THE R_A AND R_B REACTIONS CAN BE PROJECTED (DECOMPOSED) ACCORDING TO THE LINE OF THE BIRTH POINTS (AB) AND THE Z AXIS. THEY WILL BE MARKED WITH H' AND V'!
THESE R_A AND R_B REACTIONS CAN ALSO BE PROJECTED ON THE X AND Z AXIS (THEY WILL BE MARKED AS H AND V).



IN ORDER TO COMPUTE THE VERTICAL COMPONENTS OF THE REACTIONS, EQUATIONS OF BENDING MOMENTS CAN BE WRITTEN IN POINTS A AND B.

$$\sum M_B = 0 \qquad L \cdot V'_A - \sum_A^B P_i \cdot d_{iB} = 0$$

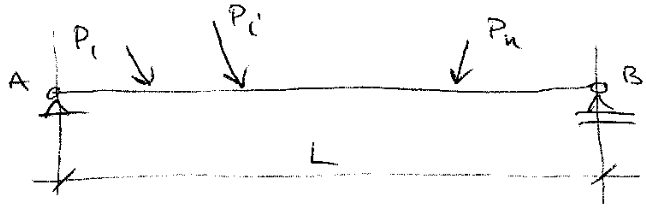
$$\sum M_A = 0 \qquad L \cdot V'_B - \sum_A^B P_i \cdot d_{iA} = 0$$

THUS WE SHALL GET:

$$V'_A = \frac{\sum_A^B P_i \cdot d_{iB}}{L} \qquad \text{AND} \qquad V'_B = \frac{\sum_A^B P_i \cdot d_{iA}}{L}$$

REMARK: IN ORDER TO COMPUTE THE VERTICAL COMPONENTS OF THE REACTIONS (USING THE Z AXIS AND THE AB LINE FOR PROJECTIONS), THE SAME PROCEDURE CAN BE APPLYED AS IN CASE OF A SIMPLE SUPPORTED BEAM.

FROM THE ABOVE REMARK RESULTS THAT IN ORDER TO OBTAIN THE VERTICAL COMPONENTS OF THE REACTIONS, A SIMPLE SUPPORTED BEAM CAN BE CONSIDERED (WITH IDENTICAL LOADS AND SPAN). THIS BEAM IS CALLED "THE ATTACHED BEAM".



IN ORDER TO GET THE VALUES OF THE LATERAL PUSHING FORCES THE STATIC EQUILIBRIUM CAN BE WRITTEN IN THE C HINGE (AT THE CROWN), SEPARATELY FOR THE LEFT AND THE RIGHT SIDE OF THE ARCH

$$\sum M_C^L = 0 \qquad V'_A \cdot x_c - H'_A \cdot f' - \sum_A^C P_i \cdot d_{iC} = 0$$

$$\sum M_c^R = 0 \quad V'_B \cdot x'_c - H'_B \cdot f' - \sum_C^B P_i \cdot d_{ic} = 0$$

IF WE REPLACE THE EXPRESSIONS OBTAINED PREVIOUSLY FOR V'_A AND V'_B IN THE ABOVE EQUATIONS, WE SHALL GET:

$$\frac{x_c}{L} \sum_A^B P_i \cdot d_{iB} - H'_A \cdot f' - \sum_A^C P_i \cdot d_{ic} = 0$$

$$\frac{x'_c}{L} \sum_A^B P_i \cdot d_{iA} - H'_B \cdot f' - \sum_C^B P_i \cdot d_{ic} = 0$$

FROM WHERE THE FOLLOWING WILL RESULT:

$$H'_A = \frac{1}{f'} \left(\frac{x_c}{L} \sum_A^B P_i \cdot d_{iB} - \sum_A^C P_i \cdot d_{ic} \right)$$

$$H'_B = \frac{1}{f'} \left(\frac{x'_c}{L} \sum_A^B P_i \cdot d_{iA} - \sum_C^B P_i \cdot d_{ic} \right)$$

NOTE, THAT FROM THE ABOVE EXPRESSIONS RESULTS, THAT THE REACTIONS ARE NOT DEPENDING FROM THE CURVATURE OF THE ARCH. THEY DEPEND ONLY ON THE POSITION OF THE 3 HINGES!

IF WE WOULD LIKE TO COMPUTE THE COMPONENTS OF THE REACTIONS PROJECTED ACCORDING TO THE HORIZONTAL AND VERTICAL AXES (ZAX), THE FOLLOWING FORMULAS CAN BE USED:

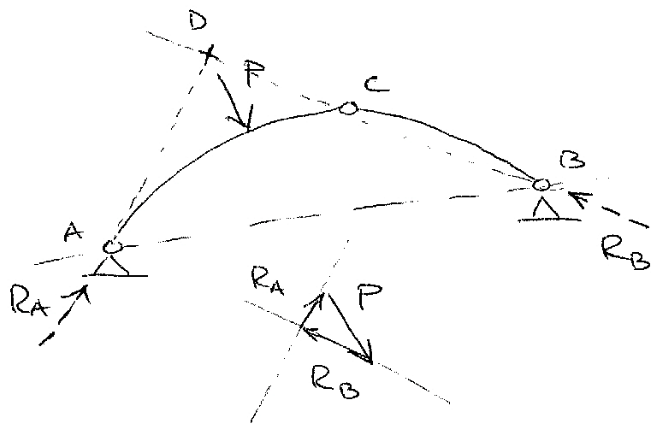
$$H_A = H'_A \cdot \cos\alpha \qquad V_A = V'_A + H'_A \cdot \sin\alpha = V'_A + H_A \cdot tg\,\alpha$$

$$H_B = H'_B \cdot \cos\alpha \qquad V_B = V'_B - H'_B \cdot \sin\alpha = V'_B - H_B \cdot tg\,\alpha$$

IN ORDER TO CHECK THE ACCURACY OF THE REACTIONS, THE UNUSED STATIC EQUILIBRIUM EQUATIONS CAN BE APPLIED:

$$\sum H = 0 \quad \text{AND} \quad \sum V = 0$$

2) GRAPHICAL SOLVING



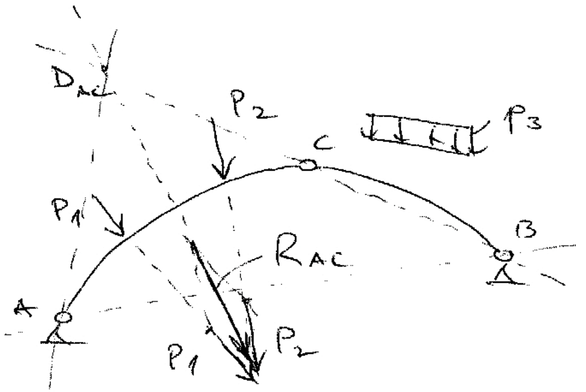
IN CASE OF ONE LOADED SIDE, DUE TO THE FACT THAT THE VALUE OF THE BENDING MOMENT IN HINGE C MUST BE ZERO (FOR BOTH PARTS OF THE ARCH, SEPARATELY), THE DIRECTION OF THE R_B REACTION WILL BE THE BC LINE.

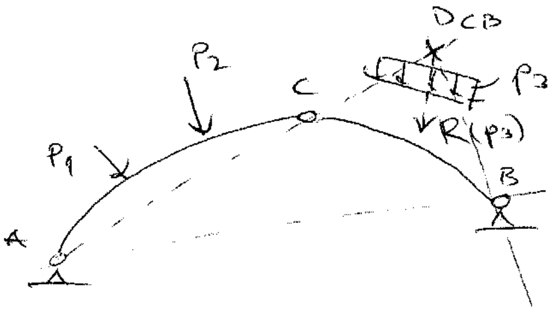
BY EXTENDING THE BC LINE, AT THE INTERSECTION WITH THE DIRECTION OF THE P FORCE WE SHALL GET A D POINT.

3 FORCES CAN BE EQUILIBRATED IN A PLANE, IF THEY ARE ALL CROSSING IN THE SAME POINT.

THUS, THE DIRECTION OF THE R_A REACTION WILL BE ACCORDING TO THE AD LINE. KNOWING THE DIRECTIONS OF THE 3 FORCES (P, R_A, R_B), THE VALUES OF THE R_A AND R_B REACTIONS CAN BE OBTAINED GRAPHICALLY (THROUGH VECTOR COMPOSITION), AS SHOWN IN THE PRECEDENT FIGURE.

THIS METHOD CAN BE USED ALSO IN CASE OF LOADS ON BOTH SIDES OF THE ARCH, CONSIDERING AT EACH STEP ONLY ONE LOADED SIDE.



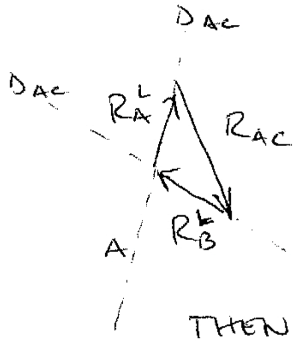


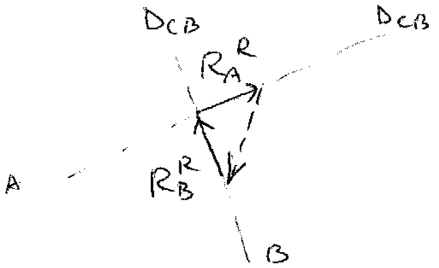
THE RESULTANT OF P_1 AND P_2 HAS TO BE FOUND (IGNORING THE RIGHT C-B SIDE AT THIS STEP)

THIS (LEFT) SIDE CAN BE SOLVED GRAPHICALLY, GETTING THE VALUES OF THE R_A^L AND R_B^L PARTIAL REACTIONS:

BY USING THE RESULTANT OF THE RIGHT SIDE LOADS (IGNORING THE LEFT A-C SIDE) THE POSITION OF POINT D_{CB} CAN BE FOUND.

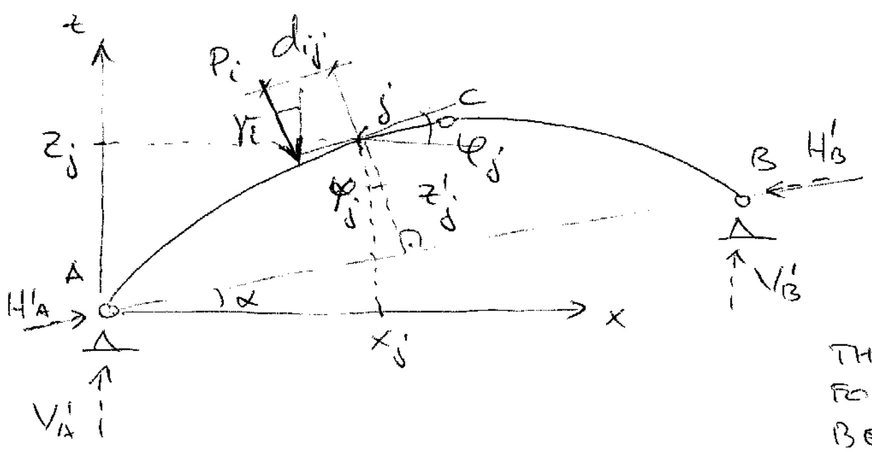
THE PREVIOUS PROCEDURE IS APPLYED FOR THIS SIDE TO!





THEN, THE PARTIAL RESULTS OBTAINED SEPARATELY CAN BE COMPOSED, OBTAINING THE COMPLETE REACTIONS GRAPHICALLY:

COMPUTING THE EFFORTS

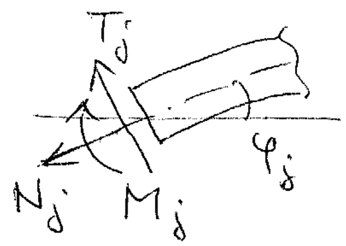


CONSIDERING A j SECTION ON THE ARCH, THE TANGENT IN THAT POINT WILL HAVE THE φ_j ANGLE (THE SAME φ_j ANGLE WILL BE BETWEEN THE VERTICAL AND z'_j).

THE DISTANCE OF THE P_i FORCE FROM SECTION j SHALL BE MARKED WITH d_{ij} AND THE ANGLE OF THE P_i FORCE TO THE VERTICAL WILL BE γ_i.

THE REACTIONS CAN BE PROJECTED ACCORDING TO THE z (VERTICAL) AXIS AND THE AB LINE (MARKED AS H' AND V').

THE EFFORTS IN SECTION j CAN BE EXHIBITED BY REMOVING THE LEFT SIDE OF THE ARCH:



BY OBSERVING THAT THE N AXIAL FORCE WILL HAVE φ_j ANGLE TO THE HORIZONTAL, THE T SHEAR FORCE WILL HAVE THE SAME φ_j ANGLE TO THE VERTICAL, THE FOLLOWING EQUATIONS CAN BE WRITTEN IN SECTION j IN ORDER TO EXPRESS THE STATIC EQUILIBRIUM:

$$N_j = -H'_A \cdot \cos(\varphi_j - \alpha) - V'_A \cdot \sin\varphi_j + \sum_A^j P_i \cdot \sin(\varphi_j - \gamma_i)$$

$$T_j = -H'_A \cdot \sin(\varphi_j - \alpha) + V'_A \cdot \cos\varphi_j - \sum_A^j P_i \cdot \cos(\varphi_j - \gamma_i)$$

$$M_j = -H'_A \cdot z'_j + V'_A \cdot x_j - \sum_A^j P_i \cdot d_{ij}$$

IN THE SAME MANNER, CONSIDERING THE ORTHOGONAL PROJECTIONS OF THE REACTIONS:

$$N_j = -H_A \cdot \cos\varphi_j - V_A \cdot \sin\varphi_j + \sum_A^j P_i \cdot \sin(\varphi_j - \gamma_i)$$

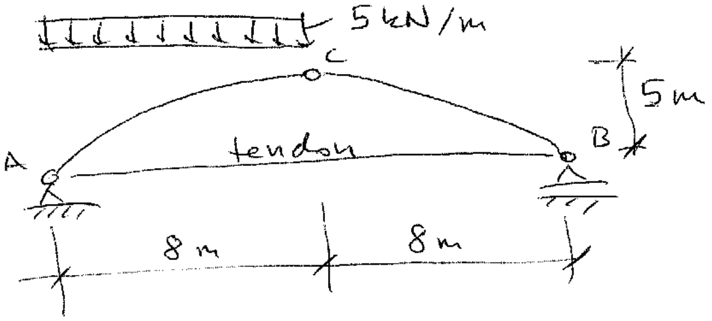
$$T_j = -H_A \cdot \sin\varphi_j + V_A \cdot \cos\varphi_j - \sum_A^j P_i \cdot \cos(\varphi_j - \gamma_i)$$

$$M_j = -H_A \cdot z_j + V_A \cdot x_j - \sum_A^j P_i \cdot d_{ij}$$

AS IT CAN BE OBSERVED IN THE ABOVE EXPRESSIONS, THE EFFORTS ARE DEPENDING ON THE SHAPE (CURVATURE) OF THE ARCH!

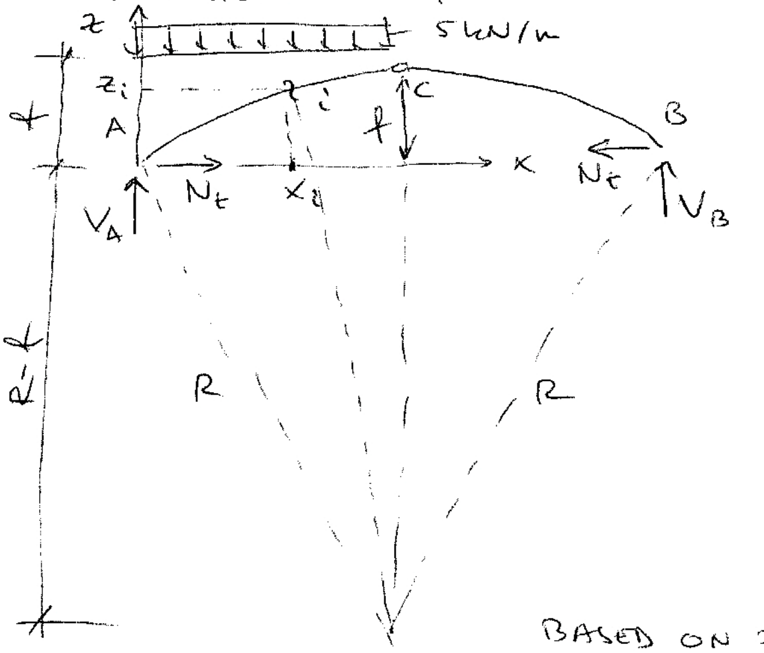
SAMPLE PROBLEM SOLVING:

GIVEN THE CIRCULAR PLANE ARCH (SHOWN IN THE PICTURE BELOW) THE DIAGRAMMS FOR N, T AND M SHOULD BE DRAWN.



FOR A PRACTICAL SOLVING THE ARCH SHALL BE DIVIDED IN SEGMENTS (IN ORDER TO COMPUTE THE EFFORTS AT EACH SECTION AT THE ENDS OF THE SEGMENTS).

THE ABOVE STATICAL SCHEME WILL BE REDRAWN, EXHIBITING THE REACTIONS (ALSO, THE RADIUS OF THE CIRCULAR ARCH)



BY CUTTING THE TENDON (THAT WAS JOINING THE A AND B POINTS), THE SAME N_t FORCE SHOUD BE PRESENT AT BOTH ENDS (IT RESULTS ALSO FROM THE HORIZONTAL EQUILIBRIUM OF FORCES).

BEEING A CIRCULAR ARCH (WITH R RADIUS), FIRST WE SHOULD ASSESS THE RADIUS, IN ORDER TO BE ABLE TO GET A FORMULA FOR THE z_i ORDINATES (FOR CHOOSEN x_i VALUES).

BASED ON PITHAGORA'S THEOREME:

$$R^2 = \left(\frac{L}{2}\right)^2 + (R-f)^2$$

THIS SHALL MEAN:

$$R^2 = 8^2 + (R-5)^2 = 64 + R^2 - 10R + 25$$

$$R = \frac{64+25}{10} = \frac{89}{10} = 8{,}9\ m$$

IN CASE OF THE i SECTION WE CAN WRITE: $\sin\varphi_i = \dfrac{\frac{L}{2} - x_i}{R} = \dfrac{8-x_i}{8{,}9}$

(SO, IF WE KNOW THE VALUE OF x_i, WE CAN GET $\sin\varphi_i$)

IN A SIMILAR WAY, $\cos\varphi_i = \dfrac{R-(f-z_i)}{R} = \dfrac{\cancel{R}}{\cancel{R}} - \dfrac{f}{R} + \dfrac{z_i}{R}$

RESULTING: $R\cdot\cos\varphi_i = R - f + z_i$ OR: $z_i = R\cdot\cos\varphi_i - R + f$

THEREFORE WE SHALL CHOOSE THE VALUES OF x_i, FROM THEM WE SHALL COMPUTE $\sin\varphi_i$, THEN $\cos\varphi_i$ (USING ARCSIN TO GET φ_i), FROM WHERE WE CAN COMPUTE THE CORRESPONDING z_i VALUES (FOR EACH AND EVERY x_i).

OF COURSE, BEFORE ALL THESE, WE HAVE TO COMPUTE THE REACTIONS.

$\sum M_B = 0 \quad V_A \cdot 16 - 5 \cdot 8 \cdot \left(\frac{8}{2} + 8\right) = 0 \quad V_A = \frac{480}{16} = 30 kN$

$\sum M_A = 0 \quad V_B \cdot 16 - 5 \cdot 8 \cdot \frac{8}{2} = 0 \quad V_B = \frac{160}{16} = 10 kN$

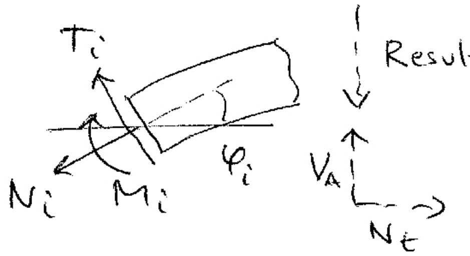
$\sum M_C^R = 0 \quad V_B \cdot 8 - N_t \cdot 5 = 0 \quad N_t = \frac{8 \cdot V_B}{5} = \frac{8 \cdot 10}{5} = 16 kN$

CHECKING THE REACTIONS: $\sum V = 0 \quad V_A + V_B = 5 \cdot 8$

$30 + 10 = 40$ ✓

$\sum H = 0 \quad N_t = -N_t$ ✓

IN ORDER TO ASSESS THE VALUES OF THE EFFORTS IN SECTIONS, THE ARCH WILL BE DIVIDED IN SEGMENTS. A TABLE WILL BE USED FOR THE CALCULATIONS AND THE EFFORTS SHALL BE COMPUTED UPON THE SCHEME BELOW:



$N_{AC} = -(V_A - p \cdot x_i) \cdot \sin\varphi_i - N_t \cdot \cos\varphi_i$

(COMPRESSION φ_i N_{AC})

$T_{AC} = (V_A - p \cdot x_i) \cdot \cos\varphi_i - N_t \cdot \sin\varphi_i$

$M_{AC} = V_A \cdot x_i - p \cdot x_i \cdot \frac{x_i}{2} - N_t \cdot z_i$

FOR THE C-B SIDE:

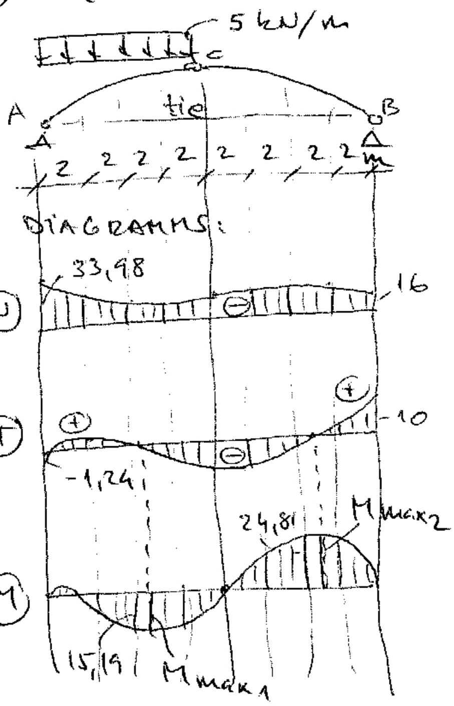
$N_{CB} = V_B \cdot \sin\varphi_i - N_t \cdot \cos\varphi_i$

$T_{CB} = -V_B \cdot \cos\varphi_i + N_t \cdot \sin\varphi_i$

$M_{CB} = V_B \cdot (L - x_i) - N_t \cdot z_i$

TABLE: [kN,m]

SECTION	x_i	$\sin\varphi_i$	$\cos\varphi_i$	z_i	N_i	T_i	M_i
A	0	0,899	0,438	0	-33,98	-1,24	0
1	2	0,674	0,739	2,67	-25,30	3,99	7,23
2	4	0,449	0,893	4,05	-18,79	1,74	15,19
3	6	0,225	0,974	4,77	-15,59	-3,60	13,64
C	8	0	1	5	-16	-10	0
4	10	-0,225	0,974	4,77	-17,87	-6,15	-16,36
5	12	-0,449	0,893	4,05	-18,79	-1,74	-24,81
6	14	-0,674	0,739	2,67	-18,56	3,40	-22,77
B	16	-0,899	0,438	0	-16	10	0



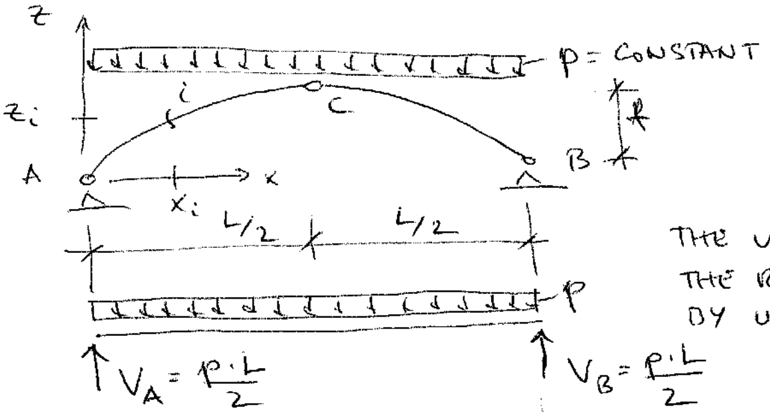
IN PRACTICE THE SEGMENTS SHOULD BE MUCH SHORTER (USALLY 30-50 CM LENGTH).

FINALLY, THE EXTREEME VALUES OF THE BENDING MOMENT SHOULD ALSO BE CALCULATED.

COINCIDENCE SHAPES

BY ARCHES, SHAPES OF COINCIDENCE ARE MEANING THOSE CURVATURES, IN CASE OF WICH THERE WILL BE ONLY AXIAL EFFORTS (NO BENDING MOMENTS AND, OF COURSE NO SHEAR FORCES). THE PROPOSED PROBLEM IS TO DETERMINE THE CORRESPONDING SHAPE OF AN ARCH FOR A GIVEN LOAD CONFIGURATION.

1) TRIPLE HINGED ARCH LOADED WITH HORIZONTALLY UNIFORMLY DISTRIBUTED VERTICAL FORCE



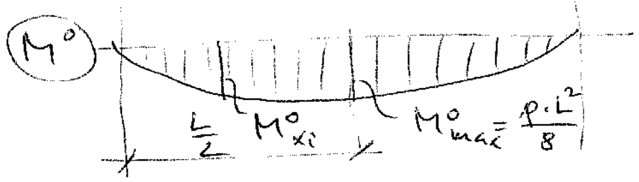
FOR A SHAPE OF COINCIDENCE:

$M_{x_i} = 0$ AND $T_{x_i} = 0$

IN EVERY i SECTION.

THE VERTICAL COMPONENTS OF THE REACTIONS WILL BE COMPUTED BY USING AN "ATTACHED BEAM".

THE BENDING MOMENT ON THE ATTACHED BEAM:



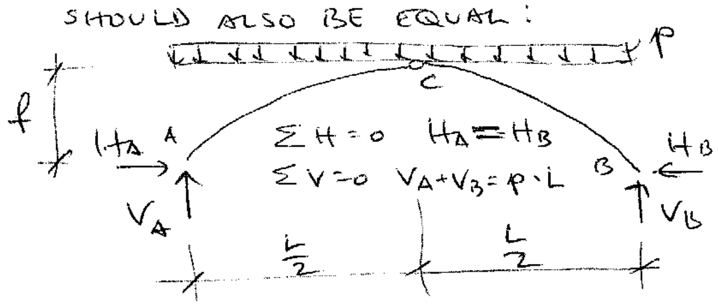
THE BENDING MOMENT IN THE ARCH (IT SHOULD BE ZERO) WILL DIFFER FROM THE ATTACHED BEAM'S DUE TO THE HORIZONTAL COMPONENTS OF THE REACTIONS:

$$M_{x_i} = M^o_{x_i} - H \cdot z_i$$

SINCE $M_{x_i} = 0$

WE CAN EXPRESS $z_i = \dfrac{M^o_{x_i}}{H}$

DUE TO THE SYMMETRY OF THE STRUCTURE, THE HORIZONTAL COMPONENTS OF THE REACTIONS SHOULD ALSO BE EQUAL:



AS WE OBSERVE, GIVEN THAT THE VALUE OF H IS CONSTANT, THE VALUES OF z_i (GIVING THE SHAPE OF THE ARCH) WILL DEPEND ON $M^o_{x_i}$ (THE DIAGRAM OF THE BENDING MOMENT IN CASE OF THE ATTACHED BEAM), WICH IS A PARABOLA.

CONSIDERING THE ATTACHED BEAM:

$$M^o_{x_i} = V_A \cdot x_i - p \cdot x_i \cdot \frac{x_i}{2} = p \cdot \frac{L}{2} \cdot x_i - p \cdot x_i \cdot \frac{x_i}{2} = p \frac{x_i}{2} (L - x_i)$$

IN ORDER TO OBTAIN H, THE STATIC EQUILIBRIUM CAN BE EXPRESSED AT THE CROWN (POINT C), FOR ONE SIDE OF THE ARCH:

$$\Sigma M^L_C = 0 \qquad V_A \cdot \frac{L}{2} - H_A \cdot f - p \cdot \frac{L}{2} \cdot \frac{L}{4} = 0$$

$$\frac{p \cdot L}{2} \cdot \frac{L}{2} - H_A \cdot f - p \cdot \frac{L^2}{8} = 0$$

THUS, WE CAN WRITE: $H_A = \frac{1}{f}\left(\frac{p \cdot L^2}{4} - \frac{p \cdot L^2}{8}\right) = \frac{p \cdot L^2}{8 \cdot f}$

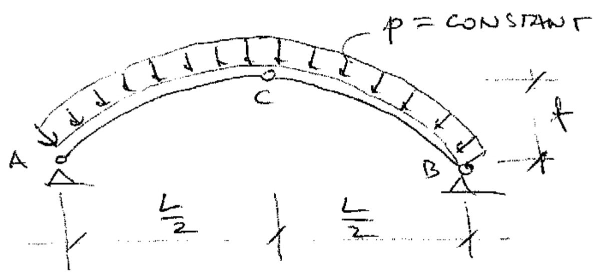
OF COURSE: $H_A = H_B = H$

BY USING THE EXPRESSION OF H IN THE FORMULA FOR z_i:

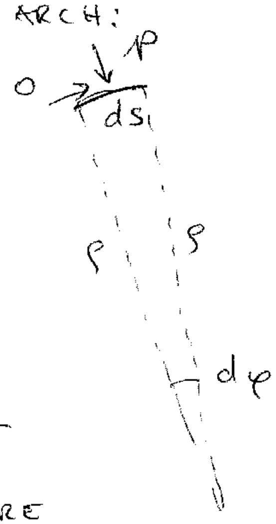
$$\boxed{z_i = \frac{M^0_{xi}}{H} = \cancel{p} \cdot \frac{x_i}{2}(L - x_i) \cdot \frac{8 \cdot f}{\cancel{p} \cdot L^2} = \frac{4 \cdot f}{L^2} \cdot x_i \cdot (L - x_i)}$$

REPRESENTING THE ECUATION OF THE SHAPE OF A PARABOLIC ARCH.

2) TRIPLE HINGED ARCH LOADED WITH UNIFORMLY DISTRIBUTED RADIAL FORCE



ACCORDING TO THE DIFFERENTIAL RELATIONS BETWEEN LOADS AND EFFORTS, WE SHALL CONSIDER A TINY CIRCULAR SEGMENT (ds) FROM THIS ARCH:



IN THIS CASE, THE NORMAL RESULTANT OF THE LOAD ON OUR TINY ds SEGMENT WILL BE p, WHILE THERE WILL BE NO TANGENT RESULTANT OF THE LOAD ($p_t = 0$)

FROM THE DIFFERENTIAL RELATIONS:

$$\frac{dN}{ds} = -p_t + \frac{T}{\rho} \quad ; \quad \frac{dT}{ds} = -p_n - \frac{N}{\rho}$$

IN OUR CASE, GIVEN THE FACT THAT THERE SHOULD BE NO BENDING MOMENT ($M = 0$), THE VARIATION OF THE SHEAR FORCE SHOULD NOT OCCUR, SO:

$$\frac{dN}{ds} = -0 + \frac{0}{\rho} = 0 \quad ; \quad \frac{dT}{ds} = -p - \frac{N}{\rho} = 0$$

FROM $\frac{dN}{ds} = 0$ RESULTS NO VARIATION OF THE AXIAL FORCE (N = CONSTANT). IF THAT IS TRUE, OBSERVING ALSO THAT p = CONSTANT, FROM $\frac{dT}{ds}$ WILL RESULT THAT ρ MUST ALSO BE CONSTANT!

THIS MEANS, THAT IN THIS CASE THE SHAPE OF COINCIDENCE IS A CIRCULAR ARCH.