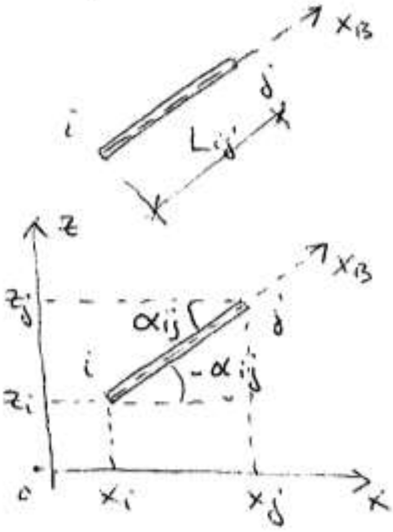
APPLYING A MATRIX FORMULATION FOR PLANAR STRUCTURES WITH PINNED JOINTS (TRUSSES)

IN FACT, THIS IS A VARIANT OF AUTOMATION OF THE NODE ISOLATION METHOD (BASED ON THE POSSIBILITY OF EXPRESSING THE STATIC EQUILIBRIUM FOR EACH NODE, THROUGH TWO EQUATIONS CONTAINING THE PROJECTED COMPONENTS OF THE FORCES).

ANY BEAM WITH PINNED ENDS (TWO-FORCE MEMBER) SHALL HAVE A LOCAL REFERENCE SYSTEM, REPRESENTED BY IT'S OWN AXIS, STARTING FROM IT'S "LEFT" END (AS ILLUSTRATED IN THE NEARBY FIGURE, WHERE THE "LEFT" END IS MARKED WITH i). TAKING INTO ACCOUNT THAT SUCH A BEAM IS A PART OF A PLANAR STRUCTURE WITH PINNED JOINTS (A PLANAR TRUSS), IT WILL ALSO BE PART OF A GENERAL REFERENCE SYSTEM (VALID FOR THE ENTIRE STRUCTURE). THUS, THE LOCAL REFERENCE SYSTEM (THE AXIS) OF THE BEAM WILL HAVE AN α_{ij} ANGLE COMPARED TO THE x AXIS OF THE GENERAL REFERENCE SYSTEM, AND THE ENDPOINTS OF THE BEAM WILL RESULT WITH PAIRS OF COORDINATES PROJECTED ON BOTH AXES OF THE GENERAL REFERENCE SYSTEM (REPRESENTING ALSO THE COORDINATES OF THE PINNED JOINTS OR NODES, CONNECTED THROUGH THE BEAM). THE ORIENTATION OF THE BEAM CAN BE EXPRESSED BY:

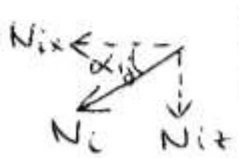


$$\cos\alpha_{ij} = \frac{x_i - x_j}{L_{ij}} \quad \text{AND} \quad \sin\alpha_{ij} = \frac{z_i - z_j}{L_{ij}}$$

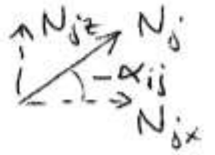
WHERE L_{ij} IS THE LENGTH OF THE BEAM (THE DISTANCE BETWEEN NODES i AND j):

$$L_{ij} = \sqrt{(x_i - x_j)^2 + (z_i - z_j)^2}$$

DUE TO THE STATIC EQUILIBRIUM, THE END FORCES OF THE BEAM WILL BE EQUAL TO EACH OTHER AND OF OPPOSITE DIRECTION. THEY CAN BE DECOMPOSED INTO PROJECTIONS ALONG THE AXES OF THE GENERAL REFERENCE SYSTEM:



$$N_{ix} = N_i \cdot \cos\alpha_{ij}$$
$$N_{iz} = N_i \cdot \sin\alpha_{ij}$$



$$N_{jx} = N_j \cdot \cos(-\alpha_{ij})$$
$$N_{jz} = N_j \cdot \sin(-\alpha_{ij})$$

THESE RELATIONS CAN ALSO BE WRITTEN IN MATRIX FORMULATION

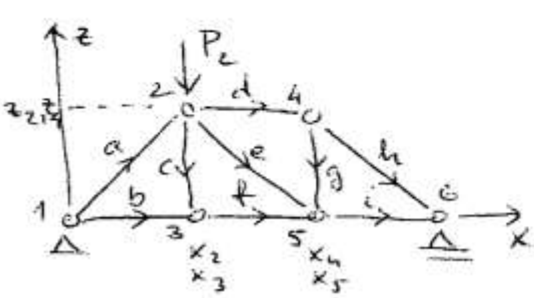
$$\begin{Bmatrix} N_{ix} \\ N_{iz} \end{Bmatrix} = N_i \begin{Bmatrix} \cos\alpha_{ij} \\ \sin\alpha_{ij} \end{Bmatrix} \quad \text{AND} \quad \begin{Bmatrix} N_{jx} \\ N_{jz} \end{Bmatrix} = N_j \cdot \begin{Bmatrix} -\cos\alpha_{ij} \\ -\sin\alpha_{ij} \end{Bmatrix}$$

RESPECTIVELY:

$$\begin{Bmatrix} N_{ix} \\ N_{iz} \end{Bmatrix} = -N_j \begin{Bmatrix} \cos\alpha_{ij} \\ \sin\alpha_{ij} \end{Bmatrix}$$

THE $\begin{Bmatrix} \cos\alpha_{ij} \\ \sin\alpha_{ij} \end{Bmatrix}$ TYPE TERMS ARE CALLED COSINE DIRECTORS, DEPICTING THE ORIENTATION OF EACH INDIVIDUAL BEAM IN A STRUCTURE.

CONSIDERING A PLANAR STRUCTURE WITH PINNED JOINTS (LIKE THE ONE ILLUSTRATED IN THE NEARBY FIGURE), AFTER MARKING WITH NUMBERS THE NODES, THE BEAMS CAN ALSO BE MARKED (WITH LETTERS) AND THE ORIENTATION OF EACH BEAM CAN BE STATED (FROM NODE i TOWARDS NODE j, WHERE $i < j$). BY KNOWING THE COORDINATES OF EVERY NODE, THE ABOVE DISCUSSED RELATIONS CAN BE WRITTEN FOR EACH INDIVIDUAL BEAM, AND THE STATIC EQUILIBRIUM CAN BE EXPRESSED ON EVERY NODE. FOR EXAMPLE, IN CASE OF NODE NUMBER 2 THE FOLLOWING EQUILIBRIUM EQUATIONS CAN BE WRITTEN:



$$\begin{cases} N_{2x}^{a} + N_{2x}^{c} + N_{2x}^{d} + N_{2x}^{e} = P_{2x} \\ N_{2z}^{a} + N_{2z}^{c} + N_{2z}^{d} + N_{2z}^{e} = P_{2z} \end{cases}$$

IN MATRIX FORMULATION THEY WILL LOOK IN THE FOLLOWING MANNER:

$$\begin{Bmatrix} N_{2x}^{a} \\ N_{2z}^{a} \end{Bmatrix} + \begin{Bmatrix} N_{2x}^{c} \\ N_{2z}^{c} \end{Bmatrix} + \begin{Bmatrix} N_{2x}^{d} \\ N_{2z}^{d} \end{Bmatrix} + \begin{Bmatrix} N_{2x}^{e} \\ N_{2z}^{e} \end{Bmatrix} = \begin{Bmatrix} P_{2x} \\ P_{2z} \end{Bmatrix}$$

USING THE RELATIONS EXPRESSED FOR A GENERIC i-j BEAM:

$$N_2^{a} \begin{Bmatrix} \cos\alpha_a \\ \sin\alpha_a \end{Bmatrix} + N_2^{c} \begin{Bmatrix} \cos\alpha_c \\ \sin\alpha_c \end{Bmatrix} + N_2^{d} \begin{Bmatrix} \cos\alpha_d \\ \sin\alpha_d \end{Bmatrix} + N_2^{e} \begin{Bmatrix} \cos\alpha_e \\ \sin\alpha_e \end{Bmatrix} = \begin{Bmatrix} P_{2x} \\ P_{2z} \end{Bmatrix}$$

AND NOTING WITH A^k THE COSINE DIRECTORS OF BEAM k:

$$N_2^{a} \cdot A^{a} + N_2^{c} \cdot A^{c} + N_2^{d} \cdot A^{d} + N_2^{e} \cdot A^{e} = P_2$$

GIVEN THAT FOR EACH AND EVERY NODE OF THE STRUCTURE CAN BE WRITTEN SIMILAR RELATIONS, IN THE END, A SYSTEM OF LINEAR EQUATIONS WILL RESULT, WHICH CAN BE EXPRESSED WITH MATRICES:

$$N \times A = P$$

IN THIS FORMULATION THE „N" COLUMN MATRIX WILL CONTAIN THE UNKNOWN VALUES OF THE EFFORTS (AXIAL FORCES IN THE BEAMS), THE „A" MATRIX WILL CONTAIN THE COEFFICIENTS OF THESE UNKNOWNS, WHIL THE „P" COLUMN MATRIX WILL CONTAIN THE PROJECTED COMPONENTS OF THE KNOWN EXTERIOR FORCES. SOLVING SUCH A SYSTEM OF EQUATIONS INVOLVES INVERTING THE A MATRIX, I.E.:

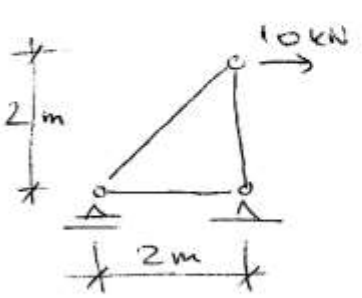
$$N = A^{-1} \times P$$

IN CASE OF CALCULATION AUTOMATION, THE NODES ARE NUMBERED IN ADVANCE, AFTER WHICH THEIR COORDINATES ARE ENTERED. FROM THE VALUES OF THE COORDINATES CAN BE COMPUTED THE LENGTHS, RESPECTIVELY THE ORIENTATIONS OF THE BEAMS, TAKING INTO ACCOUNT THEIR DIRECTION TOWARDS THE NODES (THE SIGNS OF THE $\cos\alpha$ AND $\sin\alpha$ TRIGONOMETRIC FUNCTIONS WILL RESULT DEPENDING ON THE DIRECTION OF ORIENTATION OF EACH BEAM TOWARDS THE CONSIDERED NODE).

THE REACTIONS IN THE SUPPORTS CAN BE CONSIDERED AS EXTERNAL FORCES (IF THEY WERE PREVIOUSLY CALCULATED), OR AS UNKNOWN FORCES (EFFORTS IN FICTIVE BEAMS) WITH KNOWN DIRECTIONS (ACCORDING TO THE DIRECTIONS OF THE CORRESPONDING SIMPLE LINKS FROM THE SUPPORTS).

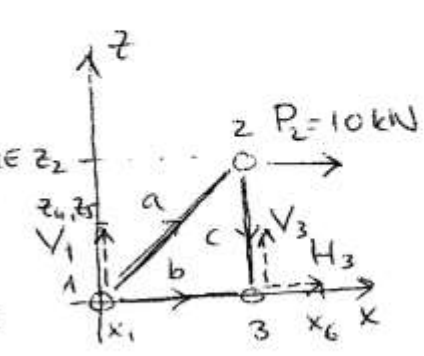
PRACTICAL EXAMPLES:

1. DETERMINE THE REACTIONS, AND THE INNER FORCES (EFFORTS IN BEAMS) FOR THE STRUCTURE SHOWED IN THE FIGURE:



SOLVING:

STARTING FROM THE STRUCTURAL SCHEME A CALCULATION SCHEME WILL BE MADE WITH THE NUMBERING OF THE NODES AND, EVENTUALLY, WITH THE NOTATION OF BEAMS (AS DISPLAYED IN THE FIGURE TO THE RIGHT).



IN THIS CASE, THE REACTIONS WERE CONSIDERED UNKNOWN EFFORTS WITH KNOWN DIRECTIONS (FICTIVE BEAMS WERE MARKED FOR THEM).

THE ORIENTATIONS OF THE DIRECTIONS OF EACH BEAM WILL

BE CONSIDERED ACCORDING TO THE NUMBERING OF THE NODES (FROM i TOWARDS j, WHERE $i < j$). THE CALCULATION OF THE COSINE DIRECTORS WILL BE DONE IN THE TABLE BELOW:

BEAM (EFFORT)	NODE i NR.	x_i	z_i	NODE j NR.	x_j	z_j	LENGTH	$\cos\alpha_{ij}$	$\sin\alpha_{ij}$
a	1	0	0	2	2	2	2,8284	-0,7071	-0,7071
b	1	0	0	3	2	0	2	-1	0
c	2	2	2	3	2	0	2	0	1
V_1	1	0	0	(4)	(0	1)	1	0	-1
V_3	3	2	0	(5)	(2	1)	1	0	-1
H_3	3	2	0	(6)	(3	0)	1	-1	0

IT CAN BE OBSERVED, THAT SUPPLEMENTERY FICTIVE NODES WERE INSERTED (4, 5 AND 6) IN ORDER TO BE ABLE TO COMPUTE THE DIRECTIONS OF THE V_1, V_3 AND H_3 UNKNOWN FORCES.

WHEN ASSEMBLING THE MATRIX „A", THE ORIENTATION OF THE BEAMS TOWARDS THE CONSIDERED NODE WILL BE TAKEN INTO ACCOUNT, AS FOLLOWS:

BEAM BETWEEN i AND j	STARTING NODE i	ENDING NODE j
	$N_i \; \alpha_{ij} \; \begin{Bmatrix} \cos\alpha_{ij} \\ \sin\alpha_{ij} \end{Bmatrix}$	$N_j \; \alpha_{ij} \; \begin{Bmatrix} -\cos\alpha_{ij} \\ -\sin\alpha_{ij} \end{Bmatrix}$

THE $N \times A = P$ MATRIX RELATION WILL SHOW AS BELOW:

$$\begin{Bmatrix} N_a \\ N_b \\ N_c \\ V_1 \\ V_3 \\ H_3 \end{Bmatrix} \times \begin{vmatrix} -0{,}707 & -1 & 0 & 0 & 0 & 0 \\ -0{,}707 & 0 & 0 & -1 & 0 & 0 \\ 0{,}707 & 0 & 0 & 0 & 0 & 0 \\ 0{,}707 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 \end{vmatrix} = \begin{Bmatrix} P_{1x} \\ P_{1z} \\ P_{2x} \\ P_{2z} \\ P_{3x} \\ P_{3z} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 10 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

WHERE THE „A" MATRIX RESULTED FROM THE COSINE DIRECTORS OF THE BEAMS AND REACTIONS, AS SHOWED BELOW:

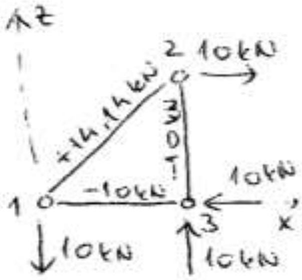
NODES \ BEAMS (FORCES)	a	b	c	V_1	V_3	H_3
1	$\{i\}$	$\{i\}$		$\{i\}$		
2	$\{j\}$		$\{i\}$			
3		$\{j\}$	$\{j\}$		$\{i\}$	$\{i\}$

IN ORDER TO INVERT THE „A" MATRIX, AN INTERNET APP WAS USED FROM HTTPS://MATRIXCALC.ORG/EN/

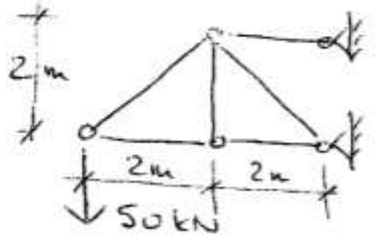
THUS, THE $A^t \times P = N$ RELATION WILL LOOK AS FOLLOWS:

$$\begin{bmatrix} 0 & 0 & 1{,}414 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 & 0 \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ 10 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 14{,}14 \\ -10 \\ -10 \\ -10 \\ 10 \\ -10 \end{Bmatrix}$$

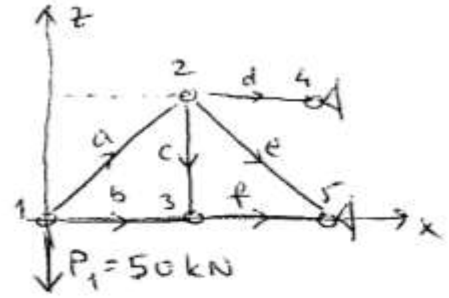
BY MARKING THE OBTAINED VALUES ON THE STRUCTURAL SCHEME, THE PROBLEM IS SOLVED: THE NEGATIVE VALUES IN THE BEAMS ARE MEANING COMPRESSION, WHILE FOR THE REACTIONS THEY MEAN CONTRARY ORIENTATION TO THE AXES OF THE GENERAL REFERENCE SYSTEM.



2. DETERMINE THE EFFORTS IN THE BEAMS OF THE STRUCTURE SHOWN IN THE FIGURE:



THE SOLVING WILL FOLLOW THE STEPS FROM THE PREVIOUS PROBLEM. IN THIS CASE, THE REACTIONS WERE NOT REQUIERED.



BEAM (EFFORT)	NODE i NR.	x_i	z_i	NODE j NR	x_j	z_j	LENGTH (m)	$\cos\alpha_{ij}$	$\sin\alpha_{ij}$
a	1	0	0	2	2	2	2,8284	-0,7071	-0,7071
b	1	0	0	3	2	0	2	-1	0
c	2	2	2	3	2	0	2	0	1
d	2	2	2	4	4	2	2	-1	0
e	2	2	2	5	4	0	2,8284	-0,7071	0,7071
f	3	2	0	5	4	0	2	-1	0

MATRIX „A":

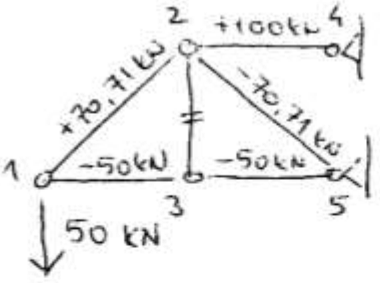
$$\begin{array}{cc|cccccc|} & & a & b & c & d & e & f \\ \hline \text{NODES} & 1 & -0{,}707 & -1 & & & & \\ & & -0{,}707 & 0 & & & & \\ & 2 & 0{,}707 & & 0 & -1 & -0{,}707 & \\ & & 0{,}707 & & 1 & 0 & 0{,}707 & \\ & 3 & & 1 & 0 & & & -1 \\ & & & 0 & -1 & & & 0 \\ \hline \end{array}$$

BEAMS (EFFORTS)

AFTER INVERTING THE „A" MATRIX (HTTPS://MATRIXCALC.ORG/EN/)

$$\begin{vmatrix} 0 & -1{,}414 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -2 & -1 & -1 & 0 & -1 \\ 0 & 1{,}414 & 0 & 1{,}414 & 0 & 1{,}414 \\ -1 & 1 & 0 & 0 & -1 & 0 \end{vmatrix} \times - \begin{Bmatrix} 0 \\ -50 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 70{,}71 \\ -50 \\ 0 \\ 100 \\ -70{,}71 \\ -50 \end{Bmatrix}$$

THE FINAL SOLUTION IS ILLUSTRATED IN THE NEARBY FIGURE. AS IT CAN BE OBSERVED, THE EFFORTS WERE CALCULATED WITHOUT COMPUTING THE REACTIONS (THESE CAN EASILY BE OBTAINED IN CASE OF NEED).



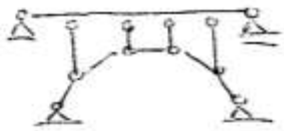
MIXED STRUCTURAL SYSTEMS

THERE ARE STRUCTURAL SCHEMES MADE BY COMBINING DIFFERENT INTERACTING SYSTEMS. THE MOST COMMON CASES ARE COUPLING BEAMS OR ARCHES WITH ARTICULATED SYSTEMS (PINNED JOINT STRUCTURES) IN CASE OF BRIDGES, TRESTLES, ETC.

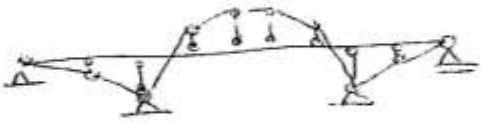
SOME EXAMPLES:

- WITH BOTTOM LOADING PATH:

- WITH UPPER LOADING PATH:



- WITH MIDDLE LOADING PATH:



USUALLY, THESE SYSTEMS ARE STATICALLY INDETERMINATE. IN ORDER TO APPROACH THEM MORE EASILY, THEIR SCHEME CAN BE TRANSFORMED IN STATICALLY DETERMINED VARIANTS (EITHER BY DECOMPOSING IN PARTS, OR BY REDUCING THE NUMBER OF THE INNER LINKS) FOLLOWING TO CAPTURE THE STRUCTURAL BEHAVIOUR. FOR EXAMPLE, IN THE CASE OF CABLE STRENGTHENED BEAMS (FIRST FIGURE), OR BEAMS STRENGTHENED BY COMPRESSED ELEMENTS (SECOND FIGURE).