## INFLUENCE DIAGRAMS

Influence diagrams are displaying the variation of an inner force in a chosen cross-section, according to the positions of a unit point load. These diagrams are usually used for mobile loads, in order to assess the most dangerous situation arising in a chosen section, but obviously, they can also be used to obtain the values of a force in a chosen cross-section. The ordinates of the diagram are also called "influence coefficients". Due to the fact that in case of static determined structures these diagrams have linear variations, they may also be called "influence lines".
In case of a simply supported straight beam (as shown in the figure below), the "E(x$_s$)" inner force is the "s" cross-section will be proportional to the "P$_i$" point loads considering the linear superposition of the effects:

$$E(x_s) = P_1 \cdot e(x_1) + \dots + P_n \cdot e(x_n)$$
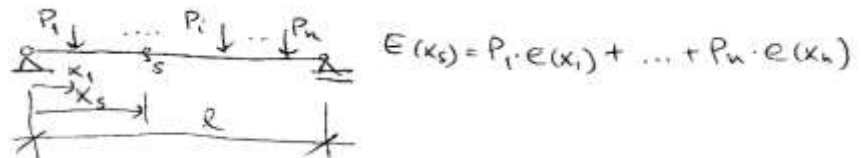
If P$_1$ = 1 and the other point loads are missing, then E(x$_s$) = 1·e(x$_1$), meaning the inner force's value in the "s" cross-section when a P$_1$ = 1 point load positioned at distance "x$_1$" from the left side is loading the beam. Decomposing in the "s" cross-section the linking force into the three known components, it can be written:

$$N(x_s) = \sum_{i=1}^{n} P_i \cdot n(x_i)$$

$$T(x_s) = \sum_{i=1}^{n} P_i \cdot t(x_i)$$

$$M(x_s) = \sum_{i=1}^{n} P_i \cdot m(x_i)$$

where n(x$_i$), t(x$_i$) and m(x$_i$) are the inner forces in section "s", corresponding to a P$_i$ = 1 load positioned at "x$_i$" distance on the beam.

The influence diagram of an inner force in a chosen cross-section can be traced either analytically, or by considering virtual displacements. The ordinate (influence coefficients) values below the reference line of the influence diagram are considered positive (marked with "+"), while those above the reference line are negative (marked with "-").

Analytical tracing:
The values of the force are computed in the chosen cross-section, considering different positions of the P =1 point load. Tracing the "m(x)" influence diagram for the marked cross-section (at distance "x" from the "A" end of a simply supported beam):

- if the P = 1 load is positioned between "A" and the cross-section:
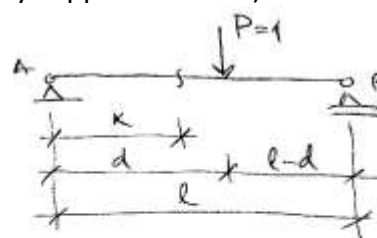
$$V_B = \frac{d}{\ell} \cdot 1$$

$$M_x = V_B \cdot (\ell - x) = \frac{d}{\ell} \cdot 1 \cdot (\ell - x)$$

$$m_x = d - \frac{d}{\ell} \cdot x$$

The above relation is the equation of a line. Considering the limit criteria:
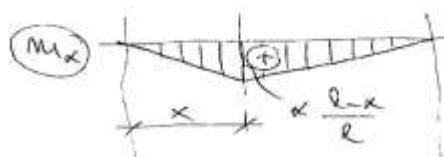
$$d = 0 \quad \rightarrow \quad m_x = 0$$

$$d = x \quad \rightarrow \quad m_x = \frac{x(\ell - x)}{\ell}$$

- if the P = 1 load is positioned between the cross-section and "B":

$$V_A = \frac{\ell - d}{\ell} \cdot 1$$

$$M_x = V_A \cdot x = \frac{\ell - d}{\ell} \cdot x$$

Considering the limit criteria:

$$d = x \quad \rightarrow \quad m_x = \frac{\ell - x}{\ell} x$$

$$d = \ell \quad \rightarrow \quad m_x = 0$$

Tracing the "t(x)" influence diagram for the marked cross-section (at distance "x" from the "A" end of a simply supported beam):

- if the P = 1 load is positioned between "A" and the cross-section:

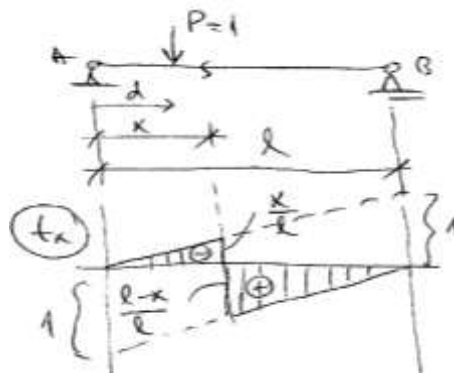$$T_x = -V_B \quad (= V_A - P)$$

$$t_x = -\frac{d}{\ell}$$

The above relation is the equation of a line. Considering the limit criteria:

$$d = 0 \quad \rightarrow \quad t_x = 0$$

$$d = x \quad \rightarrow \quad t_x = -\frac{x}{\ell}$$

- if the P = 1 load is positioned between the cross-section and "B":

$$T_x = V_A$$

$$t_x = \frac{\ell - d}{\ell}$$

Considering the limit criteria:

$$d = x \quad \rightarrow \quad t_x = \frac{\ell - x}{\ell} = 1 - \frac{x}{\ell}$$

$$d = \ell \quad \rightarrow \quad t_x = 0$$

Applying virtual displacements:

The statically determined structural scheme must be transformed in a kinematic one (representing a mechanism obtained by releasing a link and replacing with the corresponding inner force the effect of the released link). Thus, a virtual displacement can be considered, compatible with the connections of the system. Expressing the equilibrium can be done by considering the virtual mechanical work generated by the virtual movements of the P = 1 point load and the unknown cross-sectional force, according to the virtual displacement:

$$L = -E_{(x)} \cdot \Delta_{(x)} + P_i \cdot v_i = 0$$

In the above relation, "v_i" is the ordinate (influence coefficient) from the influence diagram, "E(x)" is the unknown force, while "Δ(x)" is the virtual displacement (in the direction of the "E(x)" force, but contrary as orientation).

Tracing the "m(x)" influence diagram for the marked cross-section (at distance "x" from the "A" end of a simply supported beam):
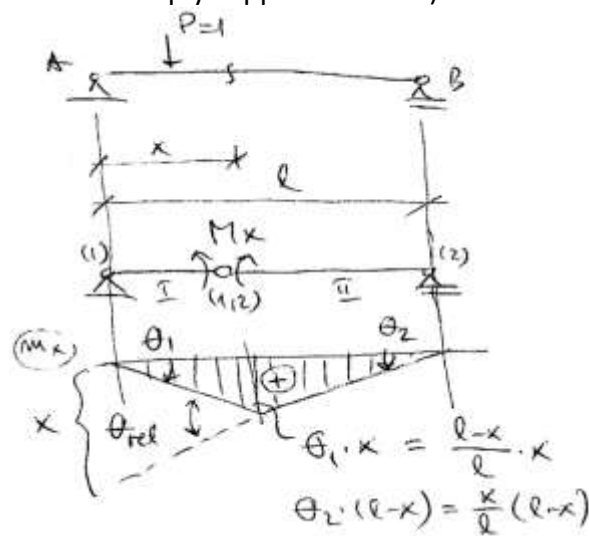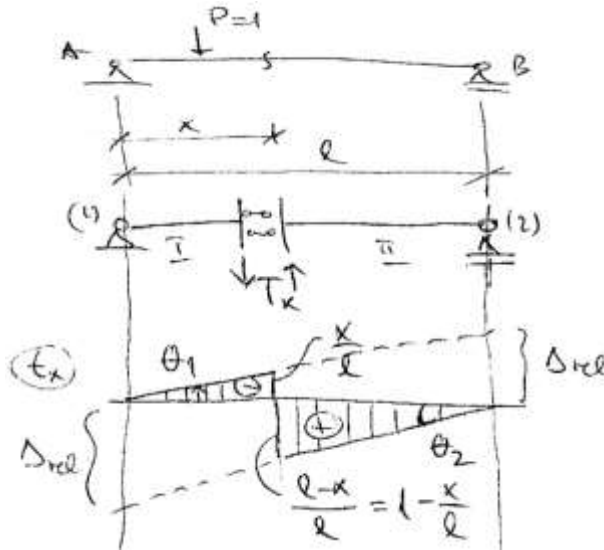

Taking into account a $\theta_{rel} = 1$ relative rotation (between the faces of the cross-section), the angle of rotation for the two bodies will be:

$$\theta_2 = \frac{x}{\ell}$$

$$\theta_1 = \frac{\ell - x}{\ell}$$

Tracing the "t(x)" influence diagram for the marked cross-section (at distance "x" from the "A" end of a simply supported beam):

Taking into account a Δ_{rel} = 1 relative displacement (between the faces of the cross-section), the angle of rotation for the two bodies will be:

$$\theta_2 = \frac{1}{\ell}$$

$$\theta_1 = \theta_2 = \frac{1}{\ell}$$

Based on these values, the ordinates of the influence diagram near the cross-section will be:

$$\theta_2 \cdot (\ell - x) = \frac{\ell - x}{\ell}$$

$$\theta_1 \cdot x = \frac{x}{\ell}$$

As a conclusion, the $E(x_s) = 1 \cdot v_i$ relation (in which 1 is the value of the mobile P point load) represents the general equation of influence diagrams. In other words, the influence diagram of an "E" inner force in the "s" cross-section (at "x_s" distance from "A") is given by the displacement diagram of the load path (line) of the system (transformed into a kinematic scheme) according to the direction of the P =1 point load, when a "$\Delta(x_s)$ = 1" virtual displacement is considered in the direction of the "$E(x_s)$" force in a way in which the "$E(x_s) \cdot \Delta(x_s)$" virtual mechanical work generated by the effort is negative.

Important note:
The ordinates in an influence diagram are showing the values of the inner force in the chosen cross-section, according to the corresponding positions of the unit point load!

USING IFLUENCE DIAGRAMS

As it was mentioned before, in case of static determined structures the influence diagrams (of reactions and internal forces) have linear variations. If the influence diagram was traced for an internal force in a cross-section, it can be used to obtain the value of that force corresponding to any load. The value of the force will be obtained by multiplying the intensity of the load with the corresponding ordinate (in case of point loads), area (in case of distributed loads), or rotation (in case of bending moments) from the influence diagram:

$$E(x_s) = P_1 \cdot v_1 + P_2 \cdot v_2 + \dots \qquad E(x_s) = p \int_a^b v_i \cdot dx = p \cdot \Omega_{ab} \qquad E(x_s) = M_i \cdot \theta_i$$
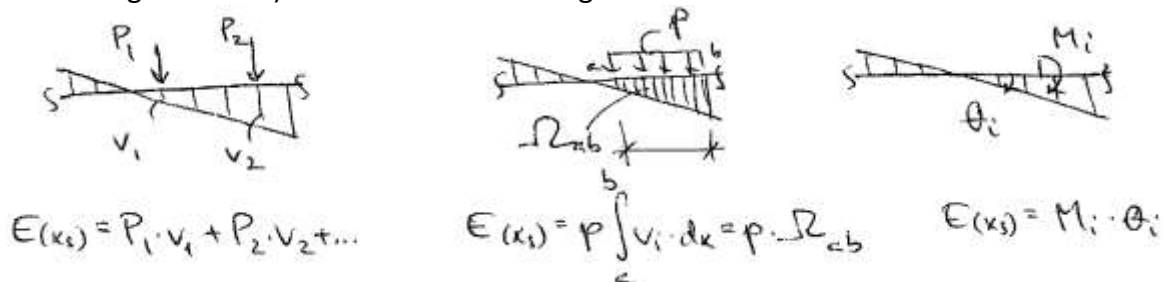
For a chosen cross-section the influence diagrams of different forces will have different shapes (the most unfavourable positions of the loads are different). The study of an inner force in several cross-sections requires tracing several influence diagrams, separately for each individual cross-section.

Numerical example: The values of "M" and "T" are required in the "i" cross-section of the below illustrated frame:
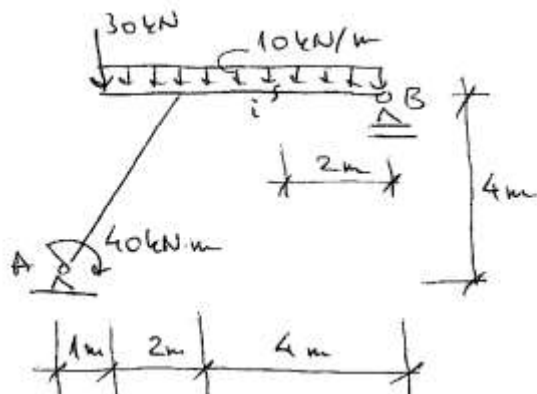

$M_i, T_i = ?$

In order to obtain "M_i" a hinge will be considered in the "i" cross-section, replacing with "M_i" the effect of the link which was released. The displacement diagram of the obtained mechanism will be traced (projected according to the directions of the loads) taking into account a θ = 1 virtual relative rotation between the faces of the "i" cross-section:

$$\theta_1 = \frac{2}{7} = 0,286$$

$$\theta_2 = \frac{\theta_1 \cdot 5}{2} = \frac{1,429}{2} = 0,714$$
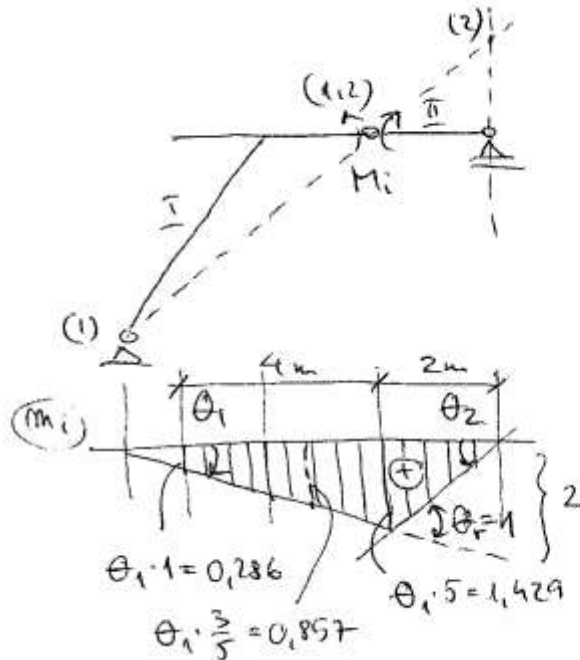
$$M_i = 40 \cdot 0,286 + 30 \cdot 0,286 +$$
$$+ 10\left(4 \cdot 0,857 + 2 \cdot \frac{1,429}{2}\right) =$$
$$= 68,59 \; kN\,m$$

In case of "T_i" the procedure is similar, considering a Δ = 1 relative displacement:

$$\theta_1 = \theta_2 \rightarrow (1,2) \; \underset{\infty}{}$$

$$\theta_1 = \frac{1}{7} = 0,143 \; ; \; \theta_2 = 0,143$$
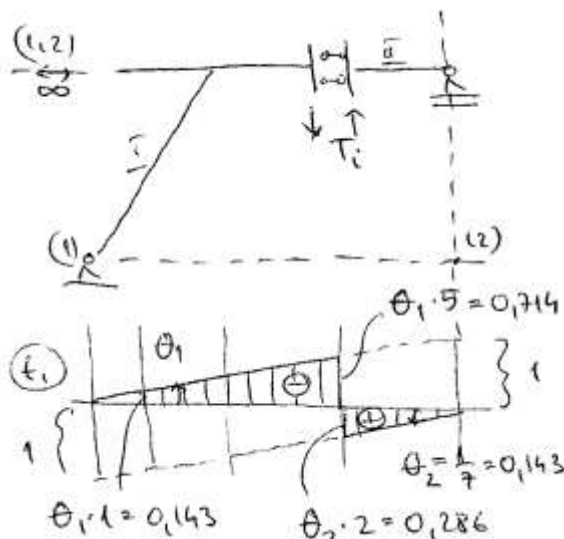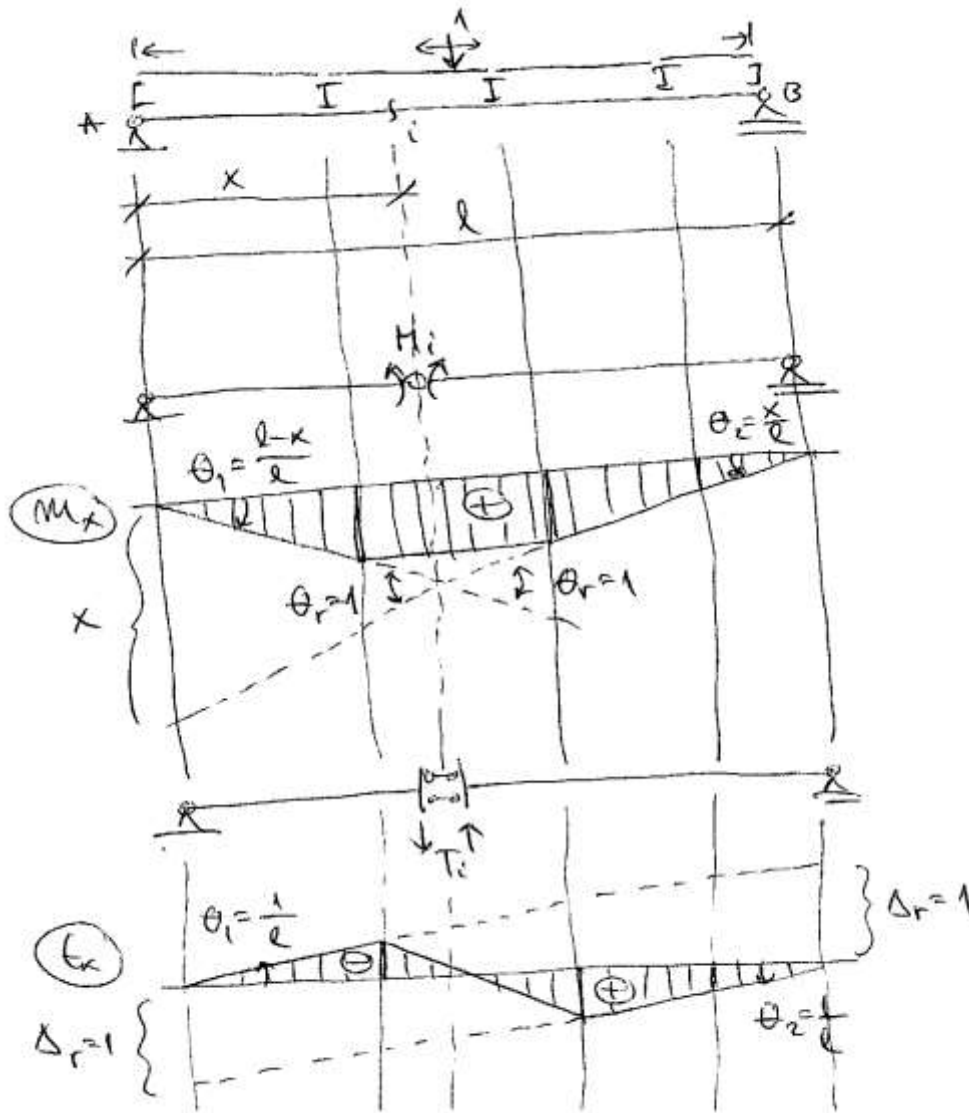
$$T_i = -40 \cdot 0,143 - 30 \cdot 0,143 -$$
$$-10\left(4 \cdot 0,143 \cdot \frac{3}{5} - 2 \cdot \frac{0,286}{2}\right) =$$
$$= -24,30 \; kN$$

INDIRECT APPLIED LOADS

In such cases, the displacement of the load path (load line) should be considered, knowing that the loads will be transferred through the contact points between the secondary and main structure. Between these contact points, the influence diagram will be linear.
Illustration:

INFLUENCE DIAGRAMS FOR PLANAR ARCHES

As mentioned before, the influence diagrams can be traced analytically (computing the values of the desired reaction or force in the chosen cross-section considering different positions of the unit point load) or through virtual displacements (transforming the structural scheme in a kinematic one, e.g. mechanism).

Analytical tracing (using the attached beam):
The influence diagrams for V_A' and V_B' will be obtained as in the case of a simply supported beam (the attached beam), while for H' the bending moment's line constructed in the "C" hinge for the attached beam will be divided by f' (f' = f·cos α, therefore H' = H / cos α):
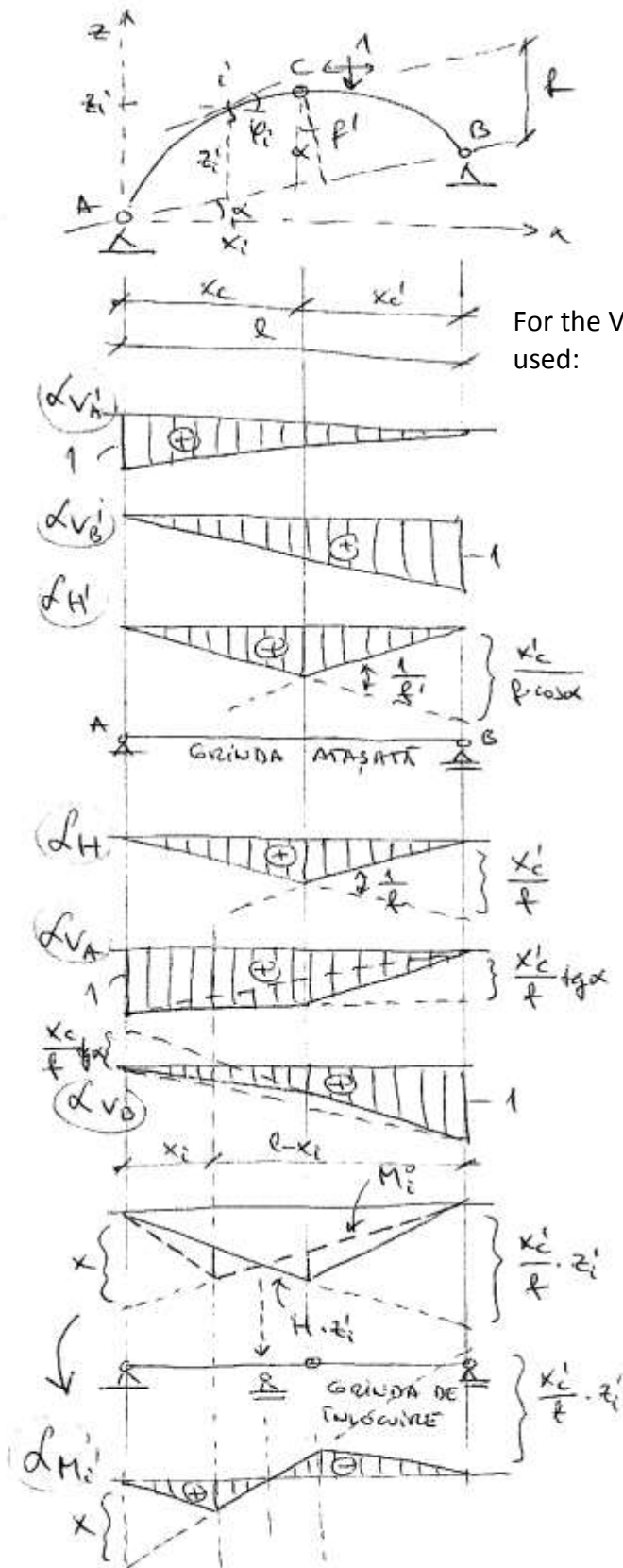
$$\ell' = \ell \cdot \cos\alpha \;\rightarrow\; H' = \frac{H}{\cos\alpha}$$

For the V_A, V_B and H diagrams these relations can be used:
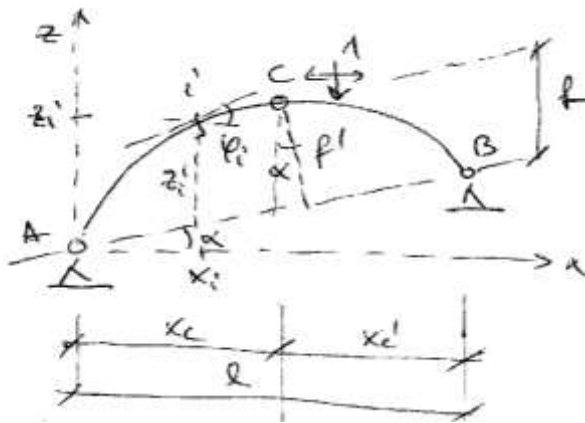
$$H = H' \cdot \cos\alpha$$

$$V_A = V_A' + \frac{M_c^o}{f} \cdot tg\,\alpha$$

$$V_B = V_B' - \frac{M_c^o}{f} \cdot tg\,\alpha$$

The diagram for H results commonly from H'

The V_A and V_B diagrams can be obtained by superimposing V_A' and V_B' with the M_c^o diagram, multiplied by $\pm$ tg α / f

For the M_i diagram, the M_i^o should be superposed with the H diagram multiplied by z_i'
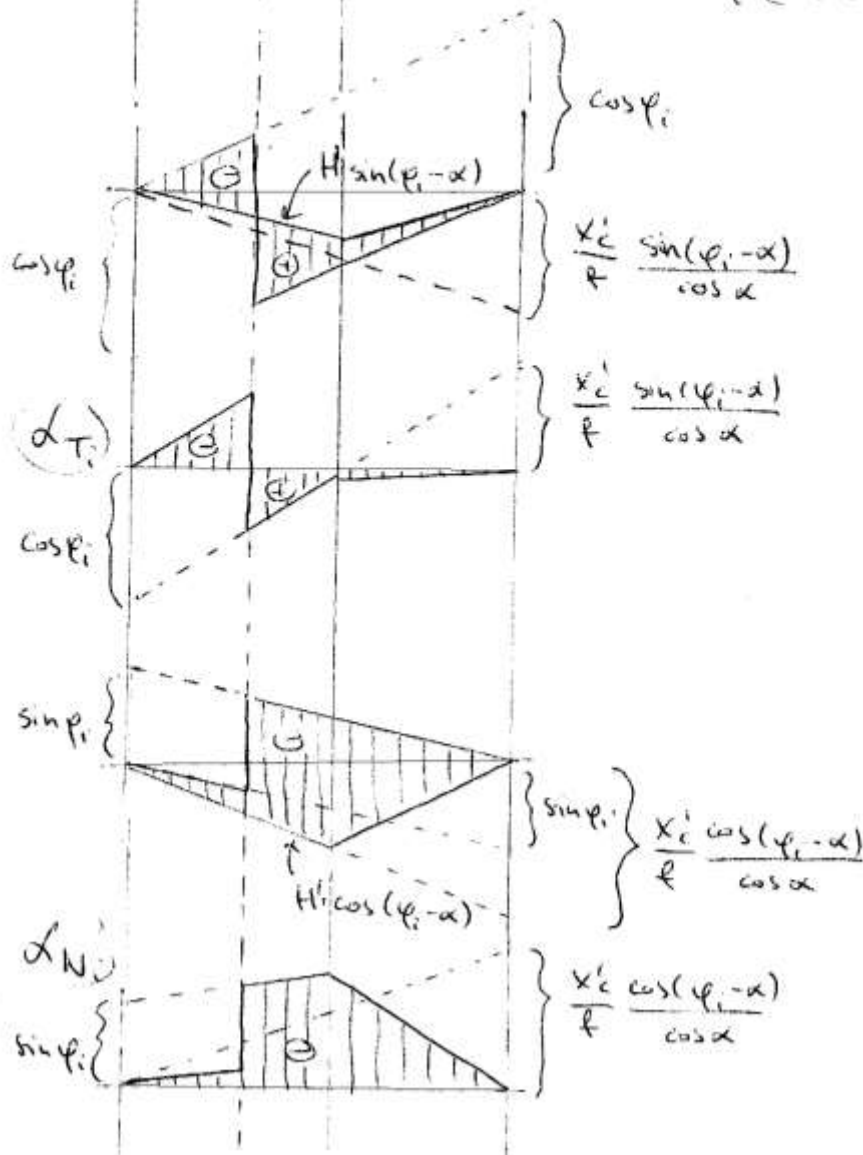
The T_i diagram can be obtained through:

$$T_i^0 \cdot \cos\varphi_i - H' \cdot \sin(\varphi_i - \alpha)$$

As for N_i :

$$T_i^0 (-\sin\varphi_i) - H' \cdot \cos(\varphi_i - \alpha)$$

Applying virtual displacements:
Simple links will be considered instead of the supports, in order to display the reactions. A mechanism composed from 2 bodies will result, and the instant rotation centres should be obtained for proper tracing of the displacement diagrams. In case of the efforts in section "i", the corresponding mechanisms will be used (replacing the released link with the link force):
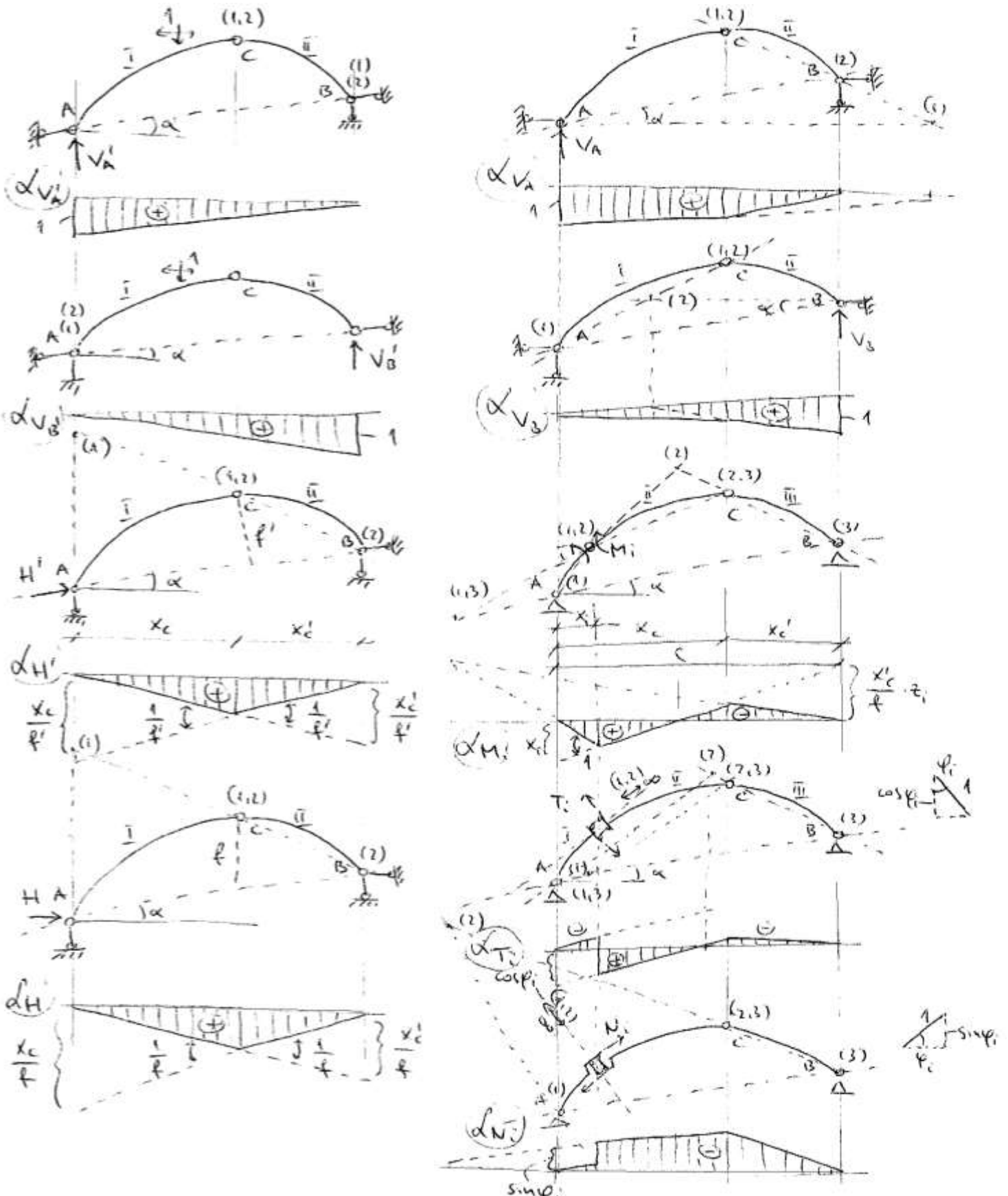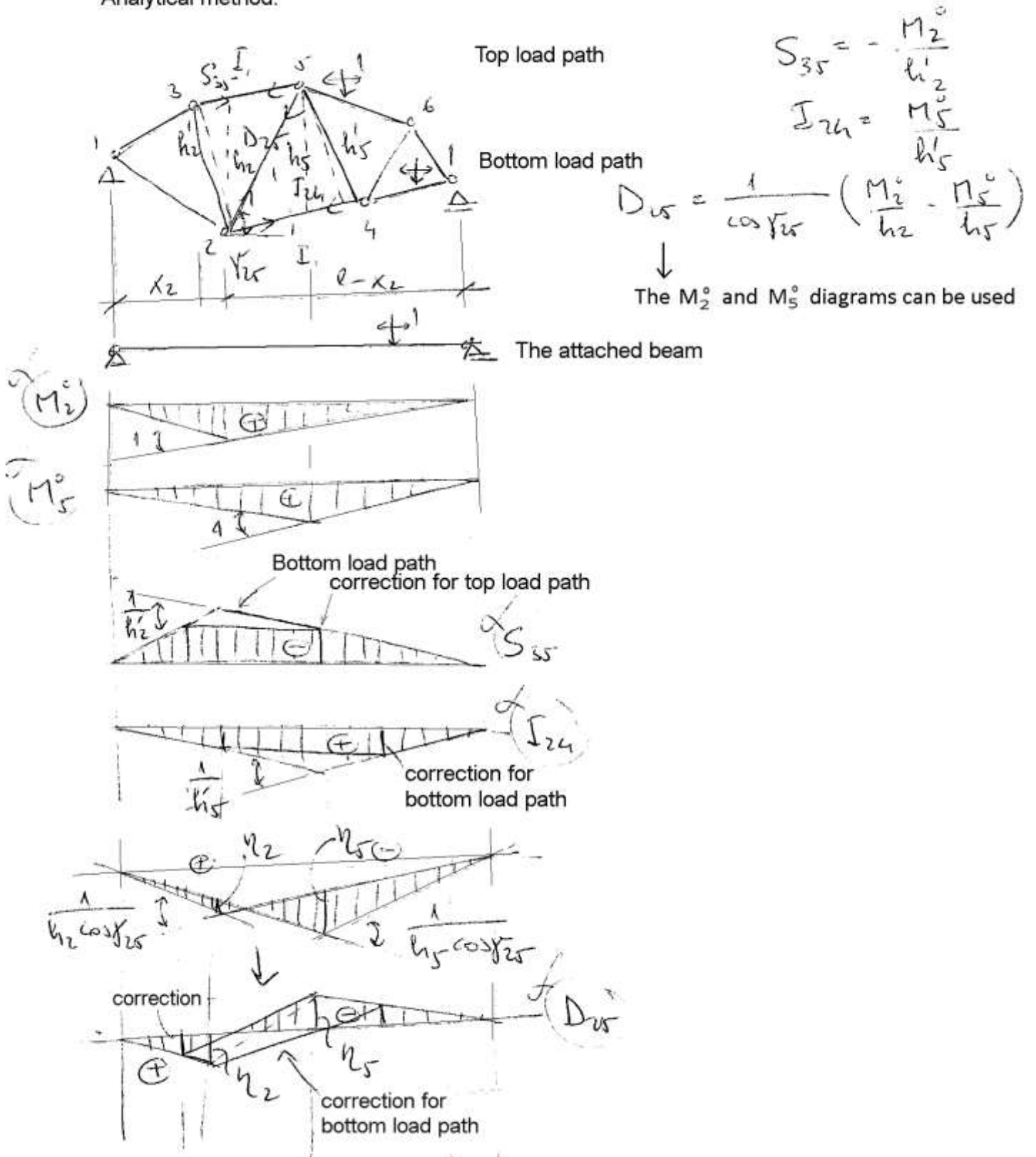
INFLUENCE DIAGRAMS FOR PLANAR PIN-JOINTED STRUCTURES (TRUSSES)

Example:
The "P =1" mobile point load can be applied either on the upper nodes (top load path), or on the lower ones (bottom load path), resulting different diagrams (the linear variation will be from node to node in both cases).
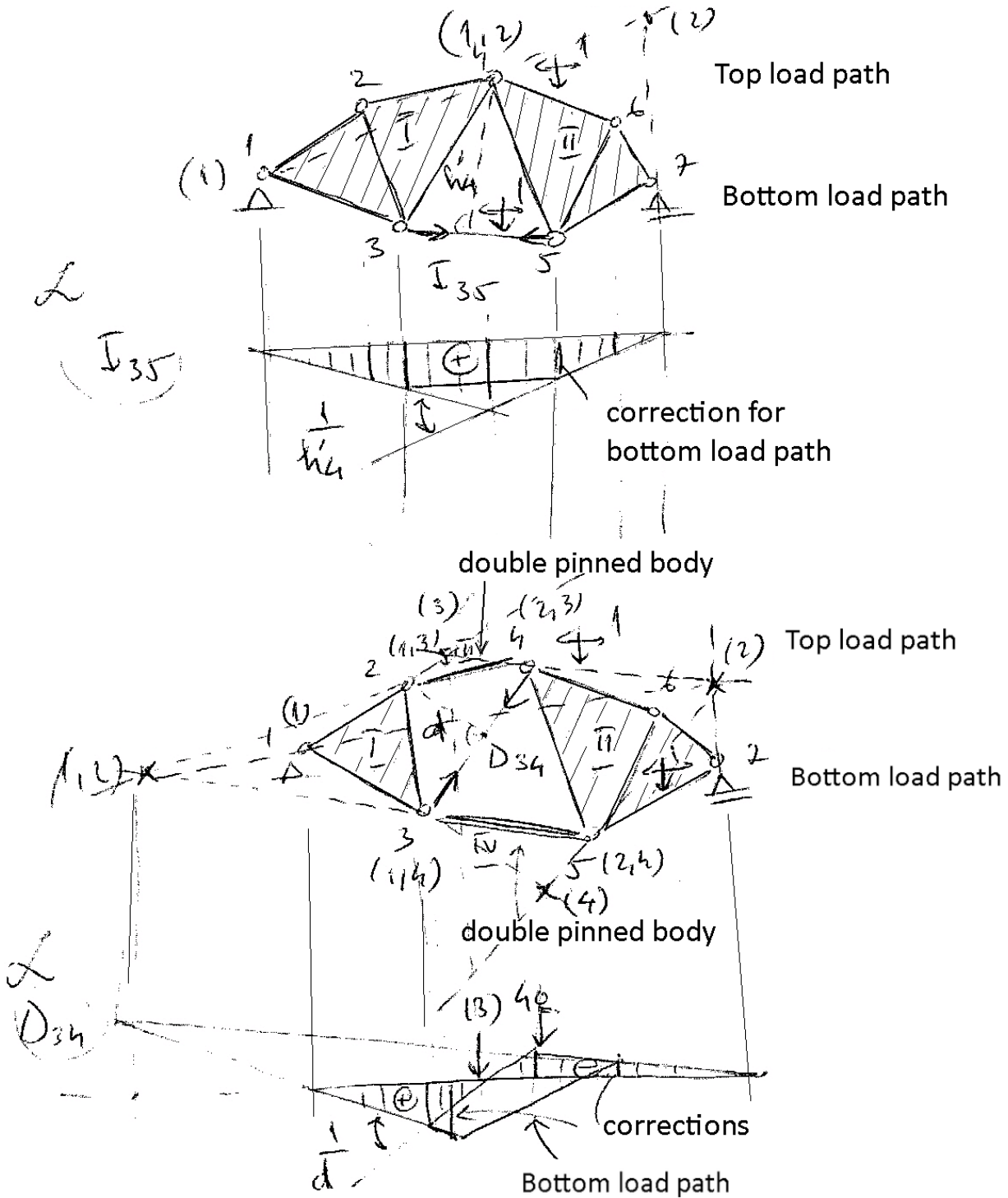
Analytical method:

Top load path

Bottom load path

$$S_{35} = - \frac{M_2^o}{h_2'}$$

$$I_{24} = \frac{M_5^o}{h_5'}$$

$$D_{25} = \frac{1}{\cos \gamma_{25}} \left(\frac{M_2^o}{h_2} - \frac{M_5^o}{h_5} \right)$$

$\downarrow$

The M_2^o and M_5^o diagrams can be used

The attached beam

(M_2^o)

(M_5^o)

Bottom load path
correction for top load path

$\frac{1}{h_2'}$ S_{35}

$\frac{1}{h_5'}$ I_{24}

correction for
bottom load path

$\frac{1}{h_2 \cos \gamma_{25}}$ $\frac{1}{h_5 \cos \gamma_{25}}$

correction D_{25}
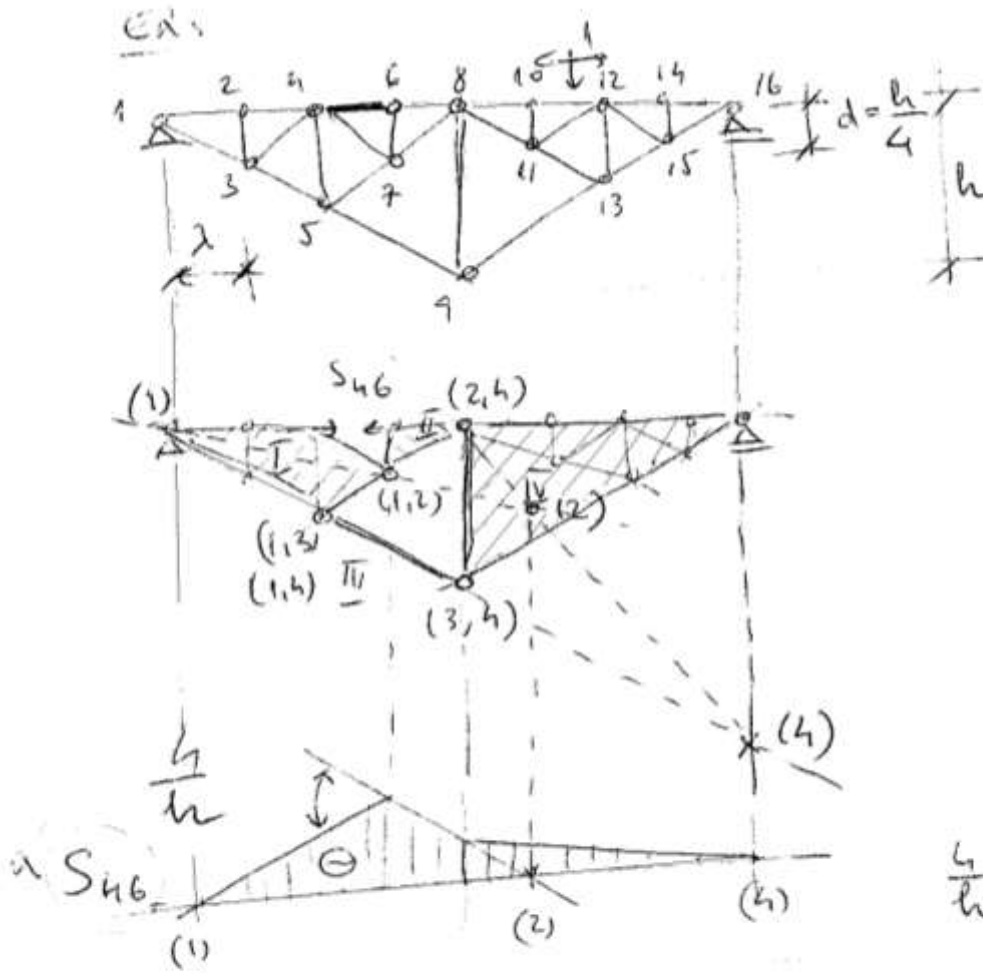
correction for
bottom load path

The statically determined structural scheme of the planar truss is transformed in a kinematic one (representing a mechanism obtained by replacing a beam with the corresponding axial force).
Thus, a unit virtual displacement can be considered along the axis of the replaced beam.

Applying virtual displacements

Top load path

Bottom load path

correction for bottom load path

double pinned body

Top load path

Bottom load path

double pinned body

corrections

Bottom load path

In case of composed structures, the structure can be considered either as a whole, or decomposed in parts.

In order to obtain the diagram for the S_{46} axial force, the beam between nodes 4 and 6 is replaced by the axial force, thus a mechanism composed of 4 bodies is obtained. After finding the positions of the instant rotation centers, a virtual displacement is given by increasing the distance between nodes 4 – 6 with 1 unit.

Or, by decomposing in parts: