ELASTIC DISPLACEMENTS

In many cases the proper design of a structure requires to take into account not only the load-bearing capacity, but also the limitation of the displacements or deformations. The calculation of elastic punctual displacements (considering elastic behaviour) of a load-bearing structure is based on the use of the principle of virtual mechanical work. For static equilibrium, the total mechanical work of the external and inner forces must be null. Virtual displacements considered in the case of virtual mechanical work must be compatible with the connections, representing a possible deformation of the structure.

The general equilibrium condition in a deformed state

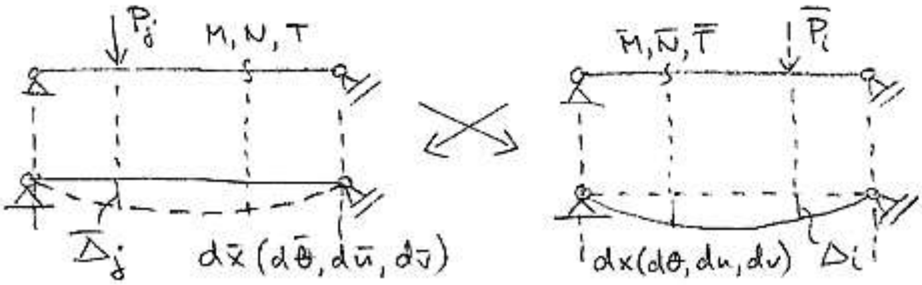
In case of a structure made out of beams, if the P_j real load travels through a Δ_j virtual displacement (the virtual terms are marked with a line above):

$$\sum P_j \cdot \bar{\Delta}_j - \int M \cdot d\bar{\theta} - \int N \cdot d\bar{u} - \int T \cdot d\bar{v} = 0$$

This is the equilibrium condition for real forces.
Considering a virtual P_j force travelling through a real Δ_j displacement:

$$\sum \bar{P}_i \cdot \Delta_i - \int \bar{M} \cdot d\theta - \int \bar{N} \cdot du - \int \bar{T} \cdot dv = 0$$

This is the compatibility condition of the real deformations.
These two relations represent the general conditions of static equilibrium of a structure. Thus, it can be concluded that in case of a straight beam the two situations illustrated in the figure below are identical:



In other words, it can be considered a real P_j load producing a virtual displacements (also real forces with virtual displacements), or a virtual P_i load corresponding to a real displacement (also virtual forces with real displacements). This will lead to the conclusion, that multiplying the force with the displacement marked on the two figures below will result in the same mechanical work.

The reciprocity theorem of mechanical work (Betti's theorem)

Due to the fact that the P load is a column matrix, using its transposed form it can be written:

$$P_j^T \cdot \bar{\Delta}_j = \int_s \sigma^T \cdot \bar{\varepsilon} \cdot ds \quad \text{and} \quad \bar{P}_i^T \cdot \Delta_i = \int_s \bar{\sigma}^T \cdot \varepsilon \cdot ds$$

These relations can further be developed taking into account the expressions of the components (already known from Strength of Materials):

$$\varepsilon_x = \frac{\sigma_x - \mu\cdot(\sigma_y + \sigma_z)}{E} \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$
$$\varepsilon_y = \frac{\sigma_y - \mu\cdot(\sigma_z + \sigma_x)}{E} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G}$$
$$\varepsilon_z = \frac{\sigma_z - \mu\cdot(\sigma_x + \sigma_y)}{E} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

So, considering real forces and stresses with virtual displacements:

$$\int_S \sigma^T \cdot \bar{\varepsilon} \cdot ds = \int_S \left(\sigma_x \cdot \bar{\varepsilon}_x + \sigma_y \cdot \bar{\varepsilon}_y + \sigma_z \cdot \bar{\varepsilon}_z + \tau_{xy} \cdot \bar{\gamma}_{xy} + \tau_{yz} \cdot \bar{\gamma}_{yz} + \tau_{zx} \cdot \bar{\gamma}_{zx}\right) ds =$$
$$= \int_S \Big[\frac{1}{E}\left(\sigma_x \cdot \bar{\sigma}_x + \sigma_y \cdot \bar{\sigma}_y + \sigma_z \cdot \bar{\sigma}_z\right) - \frac{\mu}{E}\left(\sigma_x \cdot \bar{\sigma}_y + \sigma_y \cdot \bar{\sigma}_x + \sigma_y \cdot \bar{\sigma}_z + \sigma_z \cdot \bar{\sigma}_y + \right.$$
$$\left. + \sigma_y \cdot \bar{\sigma}_z + \sigma_z \cdot \bar{\sigma}_y + \sigma_z \cdot \bar{\sigma}_x + \sigma_x \cdot \bar{\sigma}_z\right) + \frac{1}{G}\left(\tau_{xy} \cdot \bar{\tau}_{xy} + \tau_{yz} \cdot \bar{\tau}_{yz} + \right.$$
$$\left. + \tau_{zx} \cdot \bar{\tau}_{zx}\right) \Big] ds$$

The same reasoning can be done for the case of virtual forces and stresses with real displacements, leading to the conclusion that:

$$\int_S \sigma^T \cdot \bar{\varepsilon} \cdot ds = \int_S \bar{\sigma}^T \cdot \varepsilon \cdot ds$$

Meaning that the virtual mechanical work generated by real forces through virtual displacements is equal to the one generated by virtual forces through real displacements:

$$\bar{L}_{ij} = \bar{L}_{ji}$$

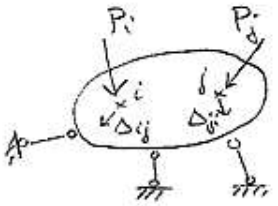
That can also be expressed as:

$P_i \cdot \bar{\Delta}_j = \bar{P}_j \cdot \Delta_i$ and $P_j \cdot \bar{\Delta}_j = \bar{P}_i \cdot \Delta_i$

Thus, the mechanical work generated by a system of P_j real forces travelling through the displacements produced by a system of P_i virtual forces is equal with the mechanical work generated by a system of P_i real forces travelling through the displacements produced by a system of P_j virtual forces.

Any structure can be loaded with forces or with displacements. Elastic displacements will occur due to force loads, while in case of displacement loads the structure will respond by showing deformation (compatible with the existing connections) and so reactions (link forces) will result.

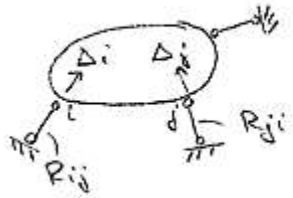
The reciprocity of unit displacements:



$$P_i \cdot \Delta_{ij} = P_j \cdot \Delta_{ji}$$
$$P_i = P_j = 1$$
$$\delta_{ij} = \delta_{ji}$$

In the above relation the first index shows the point (cross-section) of displacement and the second index the acting point of the unit load force. This relation is used in case of solving structures by the force method.

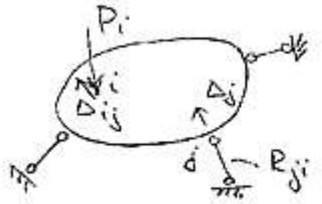
The reciprocity of unit reactions:



$$\Delta_i \cdot R_{ij} = \Delta_j \cdot R_{ji}$$
$$\Delta_i = \Delta_j = 1$$
$$r_{ij} = r_{ji}$$

The reaction in point i from loading the structure with a unit displacement in point j is equal with the reaction in point j which results from a unit displacement load applied in point i. This relation is used in case of solving structures by the displacements method.

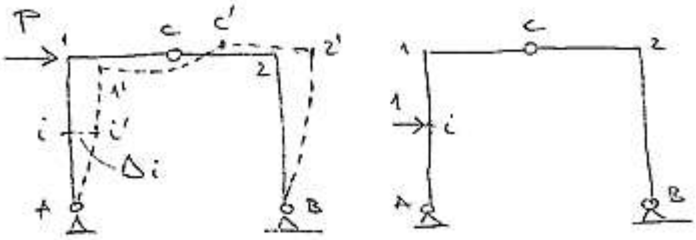
<u>The reciprocity between a unit displacement and a unit reaction:</u>



$$P_i \cdot \delta_{ij} + \Delta_j \cdot R_{ji} = 0$$
$$P_i = 1 \quad \Delta_j = 1$$
$$\delta_{ij} = -r_{ji}$$

The displacement in point i (according to the direction of $P_i = 1$) resulting from a $\Delta_j = 1$ load is equal and opposed as orientation to the reaction born in connection j, produced by $P_i = 1$. This relation is used in tracing influence diagrams for static undeterminate structures.

The general expression of punctual displacements



$dx(d\theta, du, dv)$ m_i, n_i, t_i

Considering the above frame loaded by the P force, the cross-section i will have a displacement (the horizontal component being marked as Δ_i). Thus, a virtual horizontal unit load can be considered (according to the direction of Δ_i) going through a real Δ_i displacement. Starting from the expression of elastic static equilibrium (meaning null virtual mechanical work) of the structure:

$$\sum \bar{P}_i \cdot \Delta_i - \int \bar{M} \cdot d\theta - \int \bar{N} \cdot du - \int \bar{T} \cdot dv = 0$$

For point i can be written (considering also the possible m^t torsion):

$$1 \cdot \Delta_i - \int m_i \cdot d\theta - \int n_i \cdot du - \int t_i \cdot dv - \int m^t \cdot d\varphi = 0$$

In the above relation the 1 in front is the value of the virtual unit load, which mathematically can be ignored (it is significant only as measure unit). If support displacements are also taken into account (e.g. assembling inaccuracies):

$$\Delta_{ir} = -\sum r_i \cdot \Delta_i^r$$

(Δ^r_i is the displacement of support i considering a unit r_i reaction), while in case of temperature variations between the faces of a beam or along the axis of a beam the following components will appear:

$$\alpha \cdot \frac{\Delta t}{h} \cdot dx \quad \text{and} \quad \alpha \cdot t \cdot dx$$

In the above relations α is the linear thermal expansion coefficient, Δt is the temperature difference between the faces of the cross section and h is the distance between those faces, while t is the temperature inside the beam (in case of temperature difference along the axis of the beam). In case of static determined structures the displacements of the supports and the temperature differences will not result in link forces, their effect will result only as structural geometry modification!

In case of a generalized force load that produces M_i, N_i, T_i and M_i^t link forces in the cross-section with elastic deformations, the expression of the relative displacements will be:

$$d\theta = \frac{M_i}{E\cdot I}\cdot ds \qquad du = \frac{N_i}{E\cdot A}\cdot ds \qquad dv = k\,\frac{T_i}{G\cdot A}\cdot ds \qquad d\varphi = \frac{M_i^t}{G\cdot I^t}\cdot ds$$

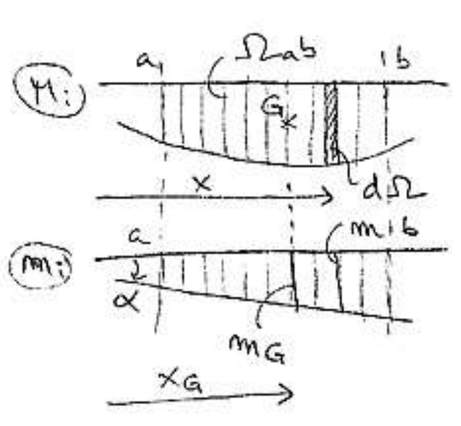
Leading to the so called "Maxwell-Mohr" relation (the general expression of displacements):

$$\Delta_i = \int_s \frac{M_i}{E\cdot I}\, m_i\cdot ds + \int_s \frac{N_i}{E\cdot A}\, n_i\cdot ds + \int_s k\cdot\frac{T_i}{G\cdot A}\, t_i\cdot ds + \int_s \frac{M_i^t}{G\cdot I^t}\cdot m_i^t\cdot ds$$
$$- \sum r_i\cdot\Delta r_i + \int_s \alpha\,\frac{\Delta t}{h}\cdot m_i\cdot ds + \int_s \alpha\cdot t\cdot n_i\cdot ds$$

This "Maxwell-Mohr" relation is applicable in case of static determinate structures, as well as in case of static indeterminate structures, as long as the deformations are elastic (reversible). The parts related to support movements and temperature differences are applicable only for static determinate structures! In case of static indeterminate structures these loads will produce link forces (and elastic deformations) due to the excess of connections, thus they should be considered together with the expressions of the inner forces from the "Maxwell-Mohr" relation.

<u>Observation</u>:

The force diagrams resulted from unit loads (m_i, n_i, t_i, m_i^t) are always linear in case of static determinate structures. The moment of inertia of the beams can be variable, if that is the case, it can be considered as a ratio of a chosen reference value (I_0).



$$\int_a^b \frac{M_i}{EI}\, m_i\cdot dx = \frac{1}{E I_0}\int_a^b \frac{I_0}{I} M_i\cdot m_i\cdot dx =$$
$$= \frac{1}{E\cdot I_0}\int_a^b m_i\left(\frac{I_0}{I} M_i\cdot dx\right) = \frac{1}{E\cdot I_0}\int_a^b m_i\cdot d\Omega =$$
$$= \frac{1}{E\cdot I_0}\int_a^b x\cdot tg\alpha\cdot d\Omega = \frac{tg\alpha}{E\cdot I_0}\int_a^b x\cdot d\Omega =$$
$$= \frac{1}{E\cdot I_0}\,\Omega_{ab}\cdot \underbrace{x_G\cdot tg\alpha}_{m_G} = \frac{1}{E\cdot I_0}\,\Omega_{ab}\cdot m_G$$

The above method of calculating the effect of an inner force by using the Ω_{ab} area of the non-linear diagram multiplied with the m_G ordinate from the linear diagram is called the rule of "Vereshchagin".

In most cases the right side terms of the "Maxwell-Mohr" relation have different relevance, so not all of them are taken into account by calculating elastic displacements:

- By straight beams and frames: $\Delta_i = \int_s \frac{M_x}{E\,I}\, m_i\, dx$

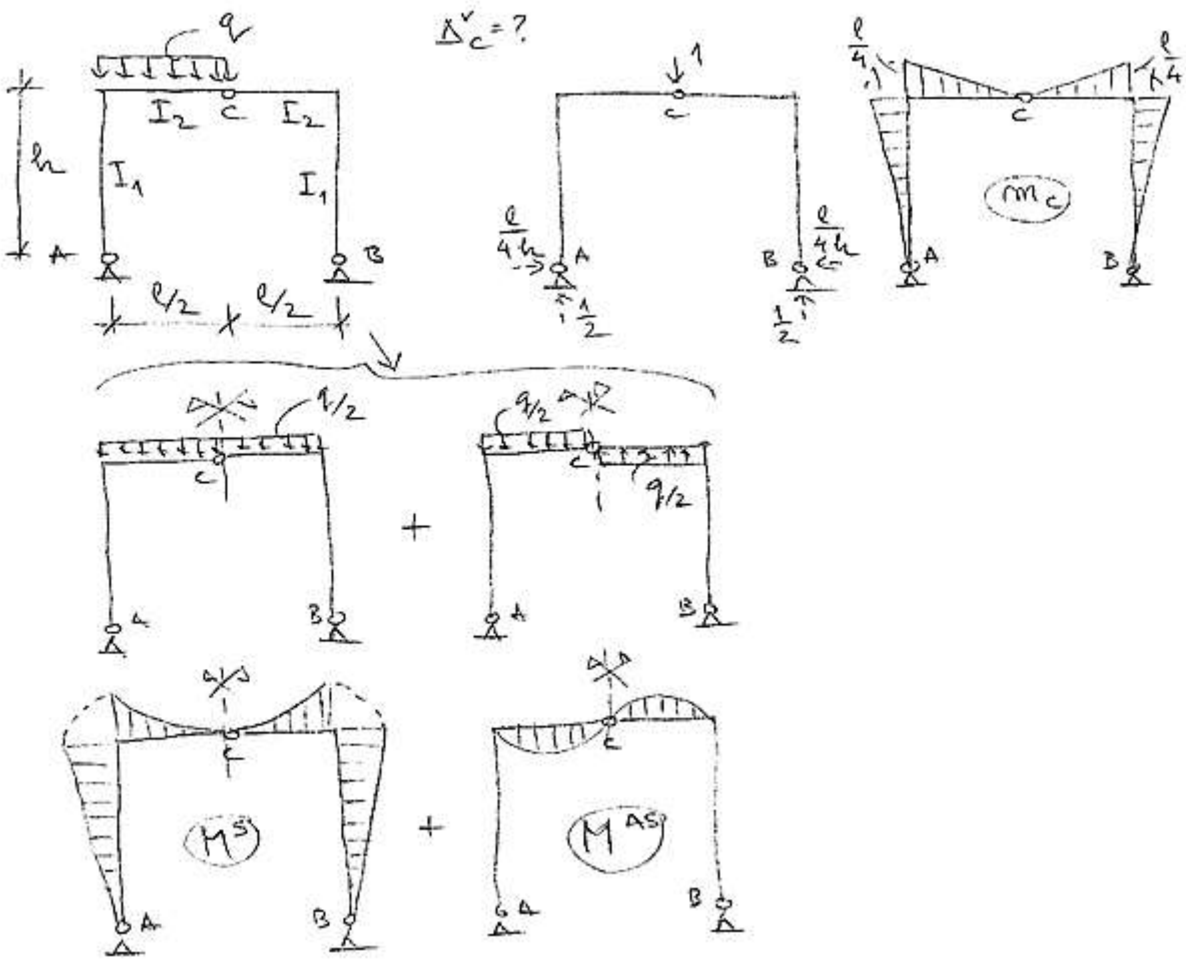
The other components have usually less than 10% influence on the values of the displacements (but in case of short beams, also the shear force is relevant).

- By trusses: $\Delta_i = \int_s \frac{N}{EA} m_i \, dx = \sum_{j=1}^{m} \frac{N_j \cdot m_{ji} \cdot l_j}{E_j \cdot A_j}$
- By arches: $\Delta_i = \int_s \frac{M}{EI} m_i \, ds + \int_s \frac{N}{EA} m_i \, ds$ if 

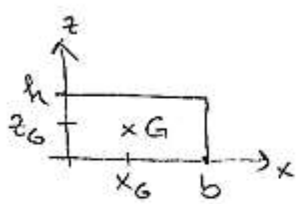
(in case of neglecting the effect of the curvature, if $\rho > 10 \cdot h$, as illustrated above).

Use of symmetry and antisymmetry

In case of the symmetrical part the calculation can be done for half of the structure (and after that multiplying by 2 the result), while in case of the antisymmetric part the diagrams will cancel each other, as showed in the following illustrating example for calculating the vertical displacement of point C:

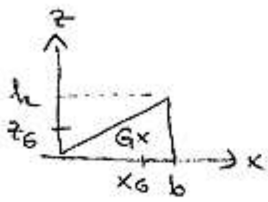


Some common geometrical characteristics (centres of gravity, surface areas):



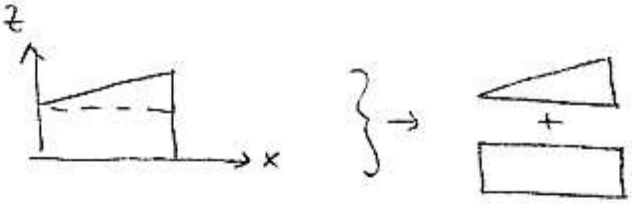
$x_G = \frac{b}{2} \quad z_G = \frac{h}{2}$

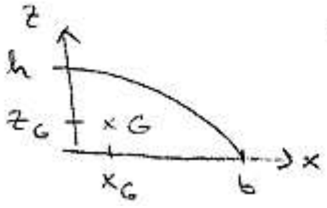
$\Omega = b \cdot h$



$x_G = \frac{2}{3} b \quad z_G = \frac{h}{3}$

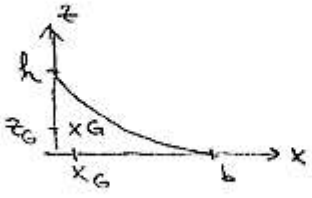
$\Omega = \frac{b \cdot h}{2}$





$x_G = \frac{3}{8} b \quad z_G = \frac{3}{8} h$

$\Omega = \frac{2}{3} b \cdot h$

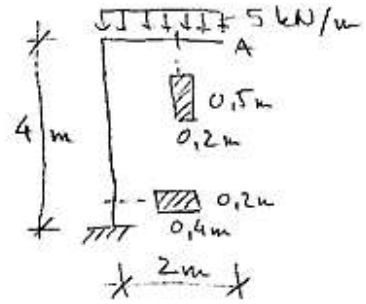


$x_G = \frac{b}{4} \quad z_G = \frac{h}{4}$

$\Omega = \frac{1}{3} b \cdot h$

Solved sample problems

1. Planar frame:



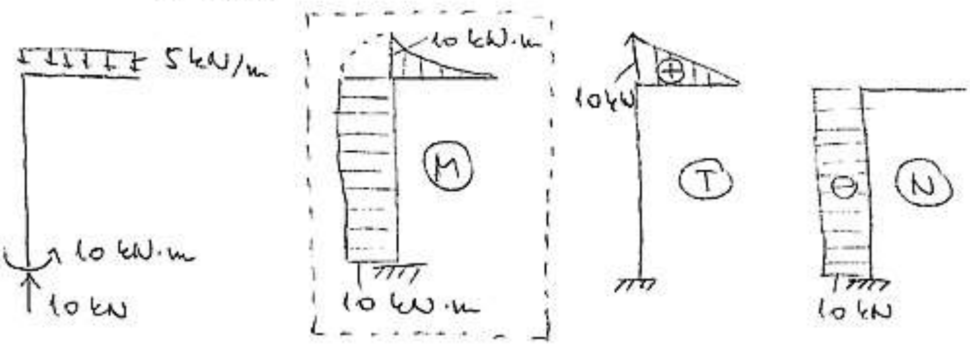
$\Delta_A^H, \Delta_A^V, \varphi_A = ?$

$I_b = \frac{b h^3}{12} = \frac{0{,}2 \cdot 0{,}5^3}{12} = 2{,}083^{-3}\ m^4$

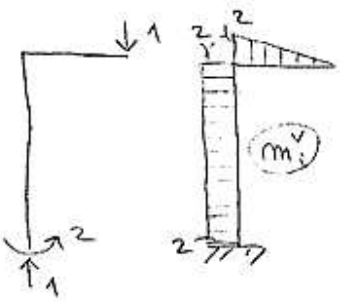
$I_c = \frac{0{,}2 \cdot 0{,}4^3}{12} = 1{,}067^{-3}\ m^4$

$I_0 = I_c \rightarrow I_b = 1{,}952 \cdot I_0$

$\Delta_i = \int \frac{M}{E \cdot I} m_i \cdot ds$



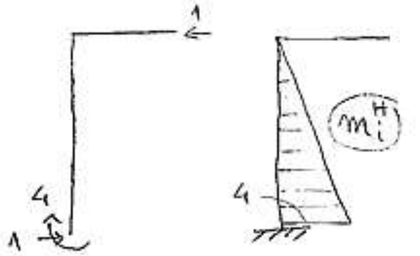
Only the M diagram will be considered due to the low influence of the other link forces.



$\Delta_A^V = \frac{1}{E I_0} \left(10 \cdot 4 \cdot 2 + \frac{1}{1{,}952} \cdot \frac{1}{3} \cdot 10 \cdot 2 \cdot \frac{3}{4} \cdot 2 \right) =$

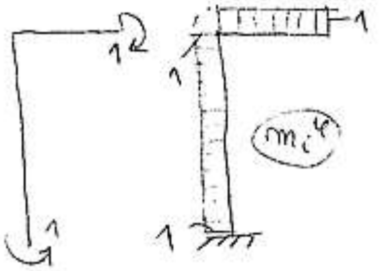
$= \frac{1}{E I_0} (80 + 5{,}128) = \frac{85{,}128}{E I_0}\ m \ \Downarrow$

As it can be observed, the vertical displacement will be downwards (in the same direction as the applied vertical unit load).



$$\Delta_A^H = \frac{1}{EI_0}\left(-10\cdot 4\cdot \frac{4}{2}\right) = -\frac{80}{EI_0}\ \text{m} \Rightarrow$$

Since the resulted value is negative, the horizontal displacement will be towards right (opposed to the direction of the applied horizontal unit load).

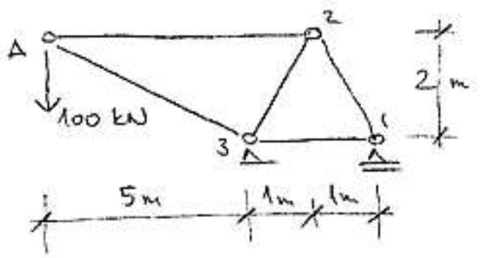


$$\varphi_A = \frac{1}{EI_0}\left(10\cdot 4\cdot 1 + \frac{1}{1{,}952}\cdot 10\cdot \frac{1}{3}\cdot 2\cdot 1\right) =$$

$$= \frac{1}{EI_0}(40 + 3{,}419) = \frac{43{,}42}{EI_0}\ \text{rad}$$

The angle of rotation will be clockwise (in the same direction with the applied unit bending moment).

2. Planar truss:

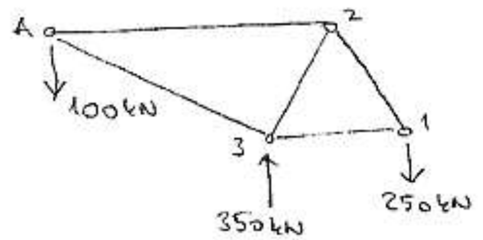


⅃L 100×100×10 STEEL

$\Delta_A^V = ?$

$A = 2\cdot 19{,}2 = 38{,}4\ \text{cm}^2 = 38{,}4\cdot 10^{-4}\ \text{m}^2$

$E = 2{,}1\cdot 10^6\ \frac{\text{daN}}{\text{cm}^2} = 2{,}1\cdot 10^8\ \frac{\text{kN}}{\text{m}^2}$

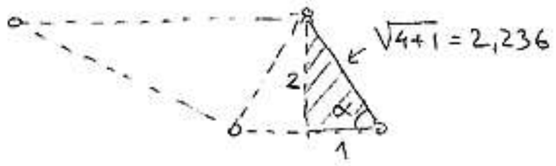


$\Sigma M_1 = 0 \quad V_3\cdot 2 = 100\cdot 7 \quad V_3 = \frac{700}{2} = 350\ \text{kN}$

$\Sigma M_3 = 0 \quad V_1\cdot 2 + 100\cdot 5 = 0 \quad V_1 = -\frac{500}{2} = -250\ \text{kN}$

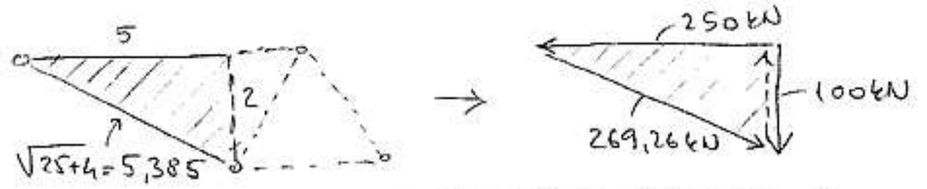
$\Sigma H = 0 \quad H_3 = 0$

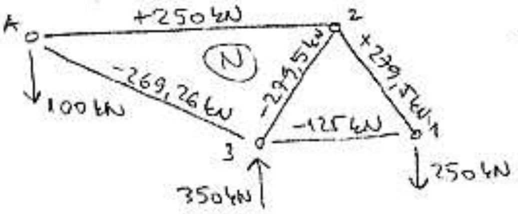
$\Sigma V = 0 \quad 350 - 100 - 250 = 0$ ✓



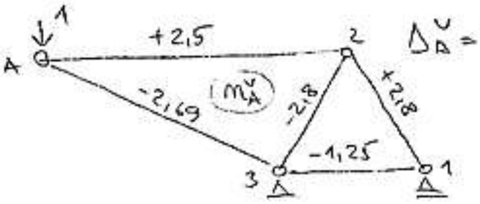
$\sin\alpha = \frac{2}{\sqrt{5}} = 0{,}894$

$\cos\alpha = \frac{1}{\sqrt{5}} = 0{,}447$





$$\Delta_A^V = \int \frac{N}{EA}\, m_i \cdot dx$$



$$\Delta_A^V = \frac{(250\cdot 6\cdot 2{,}5 + 269{,}26\cdot 5{,}385\cdot 2{,}69 + 279{,}5\cdot 2{,}236\cdot 2{,}8\cdot 2 + 125\cdot 2\cdot 1{,}25)}{2{,}1\cdot 10^5\cdot 38{,}4\cdot 10^{-4}} =$$

$$= \frac{3750 + 3900{,}26 + 3499{,}79 + 312{,}5}{80{,}64\cdot 10^4} = \frac{11462{,}55}{80{,}64\cdot 10^4} =$$

$$= 142{,}14\cdot 10^{-4}\, m = 0{,}0142\, m$$

Since the axial force diagrams are rectangular, only the values of the forces were marked on the structural schemes.