According to the above algorithm we have:

$$\mathfrak{N}(s01) = 2.5.11$$

$$\begin{cases}\chi_2(2.5.11) = 2, \ \mathrm{sg}(\chi_2(2.5.11)) = 1\\ \sigma(2.5.11,1) = K(2.5.11.3,2) = 3.5.11\end{cases}$$

$$\begin{cases}\chi_2(3.5.11) = 4, \ \mathrm{sg}(\chi_2(3.5.11)) = 1\\ \sigma(2.5.11,2) = \sigma(3.5.11,1) = K(3.5.11.7,5) = 3.7.11\end{cases}$$

$$\begin{cases}\chi_2(3.7.11) = 1, \ \mathrm{sg}(\chi_2(3.7.11)) = 1\\ \sigma(2.5.11,3) = \sigma(3.7.11,1) = K(3.7.11.3,3.7) = 3.11\end{cases}$$

$$\begin{cases}\chi_2(3.11) = 5, \ \mathrm{sg}(\chi_2(3.11)) = 1\\ \sigma(2.5.11,4) = \sigma(3.11,1) = K(3.11.7,11) = 3.7\end{cases}$$

$$\begin{cases}\chi_2(3.7) = 1, \ \mathrm{sg}(\chi_2(3,7)) = 1\\ \sigma(2.5.11,5) = \sigma(3.7,1) = K(3.7.3, \ 3.7) = 3\end{cases}$$

$$3 = \mathfrak{N}(A).$$

Hence we have

$$A \vdash_{Q_1} AB1 \vdash_{Q_5} A1 \vdash_{Q_1} AB1 \vdash_{Q_4} A01 \vdash_{Q_2} s01, \quad s01 \in L(G)$$

Received June 14, 1967

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APPROXIMATION OF FUNCTIONS BY A NEW CLASS OF LINEAR POLYNOMIAL OPERATORS

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In this paper the author introduces a new linear positive operator, of Bernstein type, and uses it in the theory of uniform approximation of functions.

§ 1. INTRODUCTION

1. This paper will be devoted to the approximation of a function $f = f(x)$ defined on the interval $[0, 1]$, by means of the new linear operator $P_m^{[\alpha]}(f(t); x) = P_m^{[\alpha]}(f; x)$, where

$$P_m^{[\alpha]}(f; x) = \sum_{k=0}^{m} w_{m,k}(x; \alpha) f\left(\frac{k}{m}\right), \tag{1.1}$$

and

$$w_{m,k}(x; \alpha) = \binom{m}{k} \frac{\prod_{\nu=0}^{k-1}(x+\nu\alpha) \prod_{\mu=0}^{m-k-1}(1-x+\mu\alpha)}{(1+\alpha)(1+2\alpha)\ldots(1+\overline{m-1}\alpha)}, \tag{1.2}$$

α being a parameter which may depend only on the natural number m.

One observes that this is a polynomial of degree m.

We first notice that for $\alpha = -\frac{1}{m}$ the operator (1.1) becomes the Lagrange interpolation polynomial corresponding to the function f and the equally spaced points $\frac{k}{m}$ $(k = \overline{0, m})$, namely

$$P_m^{[-\frac{1}{m}]}(f; x) = L_m\left(f; \ 0, \frac{1}{m}, \ldots, \frac{m}{m}; x\right) = \sum_{k=0}^{m} l_{m,k}(x) f\left(\frac{k}{m}\right),$$

where

$$l_{m,k}(x) = (-1)^{m-k} \frac{m^m}{k!\,(m-k)!} \cdot x\left(x - \frac{1}{m}\right)\ldots\left(x - \frac{k-1}{m}\right) \times$$
$$\times \left(x - \frac{k+1}{m}\right)\ldots\left(x - \frac{m}{m}\right).$$

But, as it is well known, this polynomial cannot provide a procedure which may be used in uniform approximation of every function $f \in C\ [0, 1]$.

Next we remark that in the special case $\alpha = 0$ the operator (1.1) reduces, obviously, to the classic Bernstein polynomial

$$P_m^{[0]}(f; x) = B_m(f; x) = \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} f\left(\frac{k}{m}\right). \tag{1.3}$$

It is seen that if $x \in [0, 1]$ and $\alpha \geq 0$ then we have $w_{m,k}(x; \alpha) \geq 0$ $(k = \overline{0, m})$. Hence the linear polynomial operator (1.1) is *positive* on the interval $[0, 1]$, in the sense that if on this interval we have $f \geq 0$ then we also have $P_m^{[\alpha]}(f; x) \geq 0$. As a matter of fact, *it is a polynomial of Bernstein type.*

We shall henceforth assume that the parameter α *is non-negative.*

2. Now we wish to notice that the Mirakyan [13] operator

$$M_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \tag{1.4}$$

may be obtained as a limiting case of our operator (1.1).

Indeed, let us choose $\alpha = m^{-2}$ (more generally one may take $\alpha = 0\,(m^{-1})$) and use the change of variable $x = \frac{n}{m} y$, where n is a natural number not depending on m. Then for $x = \frac{k}{m}$ we have $y = \frac{k}{n}$ and we obtain

$$P_m^{[m^{-2}]}(f; x) = \sum_{k=0}^{m} \binom{m}{k} \frac{\prod_{\nu=0}^{k-1}\left(\frac{ny}{m} + \frac{\nu}{m^2}\right) \prod_{\mu=0}^{m-k-1}\left(1 - \frac{ny}{m} + \frac{\mu}{m^2}\right)}{\left(1+\frac{1}{m^2}\right)\left(1+\frac{2}{m^2}\right)\cdots\left(1+\frac{m-1}{m^2}\right)} f\left(\frac{k}{n}\right)$$

$$= \sum_{k=0}^{m} \frac{1}{k!}\left(1-\frac{1}{m}\right)\cdots\left(1-\frac{k-1}{m}\right) \frac{\prod_{\nu=0}^{k-1}\left(ny + \frac{\nu}{m}\right) \prod_{\mu=0}^{m-k-1}\left(1 - \frac{ny}{m} + \frac{\mu}{m^2}\right)}{\left(1+\frac{1}{m^2}\right)\left(1+\frac{2}{m^2}\right)\cdots\left(1+\frac{m-1}{m^2}\right)} f\left(\frac{k}{n}\right).$$

Now we proceed to find the limit of this expression as $m \to \infty$. First of all we notice that

$$\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdots\left(1-\frac{k-1}{m}\right) \to 1, \quad \prod_{\nu=0}^{k-1}\left(ny + \frac{\nu}{m}\right) \to (ny)^k. \tag{1.5}$$

In order to find the limit of

$$E_1(m) = \prod_{\mu=0}^{m-k-1}\left(1 - \frac{ny}{m} + \frac{\mu}{m^2}\right) = \frac{\prod_{\mu=0}^{m}\left[1 + \frac{1}{m}\left(\frac{\mu}{m} - ny\right)\right]}{\prod_{\mu=m-k}^{m}\left[1 + \frac{1}{m}\left(\frac{\mu}{m} - ny\right)\right]} \tag{1.6}$$

and

$$E_2(m) = \prod_{\nu=0}^{m-1}\left(1 + \frac{\nu}{m^2}\right) = \frac{1}{1+\frac{1}{m}} \prod_{\nu=0}^{m}\left(1 + \frac{\nu}{m}\cdot\frac{1}{m}\right), \tag{1.7}$$

we shall make use of the following result (see [15]) : if f has a proper integral on the interval $[a, b]$, then

$$\lim_{m\to\infty} \prod_{j=1}^{m} (1 + f_{jm}\,\delta_m) = \exp\left(\int_a^b f(x)\,dx\right),$$

where

$$\delta_m = \frac{b-a}{m}, \quad f_{jm} = f(a + j\delta_m).$$

Setting $f = x - ny$, $a = 0$ and $b = 1$ we have $\delta_m = \frac{1}{m}$ and when $m \to \infty$ we find

$$\left(1 - \frac{ny}{m}\right)\prod_{\mu=1}^{m}\left[1 + \frac{1}{m}\left(\frac{\mu}{m} - ny\right)\right] \to \exp\left(\int_0^1 (x - ny)\,dx\right) = e^{\frac{1}{2} - ny}$$

and

$$\prod_{\mu=m-k}^{m}\left[1 + \frac{1}{m}\left(\frac{\mu}{m} - ny\right)\right] = \prod_{\nu=0}^{k}\left[1 + \frac{1}{m}\left(\frac{m-k+\nu}{m} - ny\right)\right] \to 1.$$

Therefore we have

$$E_1(m) \to e^{\frac{1}{2} - ny}. \tag{1.8}$$

Similarly, if we select $f = x$, $a = 0$ and $b = 1$, then we get

$$E_2(m) \to \exp\left(\int_0^1 x\,dx\right) = e^{\frac{1}{2}}. \tag{1.9}$$

Denoting again the variable by x we obtain just the operator (1.4) of Mirakyan, which also has been investigated by Favard [7], Szasz [19], Butzer [4], Baskakov [2], Lupaş [11], [12] and Müller [14].

3. Presupposing that $\alpha \geq 0$ has a fixed value, in Section 2 of the present paper one establishes a monotonicity property of the sequence of operators $\{P_m^{[\alpha]}(f; x)\}$, when f is convex of first order on $[0, 1]$. In Section 3 one gives a representation of the operator (1.1) by means of finite and divided differences (formulas (3.4) and (3.8)). By using the identities (4.1) and a known theorem of Bohman-Korovkin (see, for instance, Cheney [6]) one proves immediately, in Section 4, that if $x \in [0,1]$ and $0 \leqslant \alpha = \alpha(m) \to 0$ as $m \to \infty$, then the sequence of operators (1.1) converges uniformly to $f \in C\,[0,1]$. In Section 5 is estimated the corresponding order of approximation, while in Section 6 is given an expression for the remainder of approximation formula (6.1) of f by the operator (1.1). In the particular case $f \in C^1[0, 1]$ one obtains the inequality (5.6). Finally one gives an asymptotic estimate of the remainder of approximation formula (6.1).

§ 2. MONOTONICITY PROPERTIES OF THE SEQUENCE $\{P_m^{[\alpha]}(f; x)\}$

4. Assuming that the parameter α has a fixed non-negative value in each term of the sequence $\{P_m^{[\alpha]}(f; x)\}$, we shall establish here an important relationship between two consecutive terms of this sequence, which will be useful for proving a monotonicity property of it in the case of convex (or concave) functions of first order.

THEOREM 2.1. *If α is a fixed non-negative number, then between the polynomials $P_m^{[\alpha]}(f; x)$, $P_{m+1}^{[\alpha]}(f; x)$ there exists the following relationship*

$$P_{m+1}^{[\alpha]}(f; x) - P_m^{[\alpha]}(f; x) = \tag{2.1}$$

$$- \frac{1}{m(m+1)} \sum_{\nu=0}^{m-1} \frac{(x + \nu\alpha)(1 - x + \overline{m-\nu-1}\alpha)}{(1 + m\alpha)(1 + \overline{m-1}\alpha)} w_{m-1,\nu}(x; \alpha) \times$$

$$\times \left[\frac{\nu}{m}, \frac{\nu+1}{m+1}, \frac{\nu+1}{m}; f\right],$$

where $\left[\frac{\nu}{m}, \frac{\nu+1}{m+1}, \frac{\nu+1}{m}; f\right]$ *represents the divided difference of the function f on the indicated nodes.*

Proof. Using the identity $1 + m\alpha = (x + k\alpha) + (1 - x + \overline{m-k}\alpha)$ we can write

$$P_m^{[\alpha]}(f; x) = \frac{1}{1 + m\alpha} \sum_{k=0}^{m} (x + k\alpha) w_{m,k}(x; \alpha) f\left(\frac{k}{m}\right)$$

$$+ \frac{1}{1 + m\alpha} \sum_{k=0}^{m} (1 - x + \overline{m-k}\alpha) w_{m,k}(x; \alpha) f\left(\frac{k}{m}\right).$$

If in the first sum we make the change $k = i - 1$ and then denote again the summation variable by k we obtain

$$P_m^{[\alpha]}(f; x) = \frac{1}{1 + m\alpha} \sum_{k=1}^{m+1} (x + \overline{k-1}\alpha) w_{m,k-1}(x; \alpha) f\left(\frac{k-1}{m}\right)$$

$$+ \frac{1}{1 + m\alpha} \sum_{k=0}^{m} (1 - x + \overline{m-k}\alpha) w_{m,k}(x; \alpha) f\left(\frac{k}{m}\right).$$

Detaching from the first sum the term corresponding to $k = m+1$ and from the second sum the term corresponding to $k=0$ we have

$$P_m^{[\alpha]}(f; x) = \frac{1}{1 + m\alpha} \sum_{k=1}^{m} (x + \overline{k-1}\alpha) w_{m,k-1}(x; \alpha) f\left(\frac{k-1}{m}\right)$$

$$+ \frac{x(x+\alpha) \ldots (x + m\alpha)}{(1+\alpha)(1+2\alpha) \ldots (1+m\alpha)} f(1)$$

$$+ \frac{1}{1 + m\alpha} \sum_{k=1}^{m} (1 - x + \overline{m-k}\alpha) w_{m,k}(x; \alpha) f\left(\frac{k}{m}\right)$$

$$+ \frac{(1-x)(1-x+\alpha) \ldots (1-x+m\alpha)}{(1+\alpha)(1+2\alpha) \ldots (1+m\alpha)} f(0),$$

while if we detach from $P_{m+1}^{[\alpha]}(f; x)$ the two terms which have $f(0)$ and $f(1)$ as factors, then we obtain

$$P_{m+1}^{[\alpha]}(f; x) = \sum_{k=1}^{m} w_{m+1,k}(x; \alpha) f\left(\frac{k}{m+1}\right)$$

$$+ \frac{(1-x)(1-x+\alpha) \ldots (1-x+m\alpha)}{(1+\alpha)(1+2\alpha) \ldots (1+m\alpha)} f(0)$$

$$+ \frac{x(x+\alpha) \ldots (x+m\alpha)}{(1+\alpha)(1+2\alpha) \ldots (1+m\alpha)} f(1).$$

Taking these last two equalities into account we may write

$$P_{m+1}^{[\alpha]}(f; x) - P_m^{[\alpha]}(f; x) =$$

$$\sum_{k=1}^{m} \binom{m+1}{k} \frac{x(x+\alpha) \ldots (x+\overline{k-1}\alpha)(1-x)(1-x+\alpha) \ldots (1-x+\overline{m-k}\alpha)}{(1+\alpha)(1+2\alpha) \ldots (1+m\alpha)} f\left(\frac{k}{m+1}\right)$$

$$- \sum_{k=1}^{m} \binom{m}{k-1} \frac{x(x+\alpha) \ldots (x+\overline{k-1}\alpha)(1-x)(1-x+\alpha) \ldots (1-x+\overline{m-k}\alpha)}{(1+\alpha)(1+2\alpha) \ldots (1+m\alpha)} f\left(\frac{k-1}{m}\right)$$

$$- \sum_{k=1}^{m} \binom{m}{k} \frac{x(x+\alpha) \ldots (x+\overline{k-1}\alpha)(1-x)(1-x+\alpha) \ldots (1-x+\overline{m-k}\alpha)}{(1+\alpha)(1+2\alpha) \ldots (1+m\alpha)} f\left(\frac{k}{m}\right).$$

Since

$$\binom{m+1}{k} = \frac{m+1}{m+1-k} \binom{m}{k}, \quad \binom{m}{k-1} = \frac{k}{m+1-k} \binom{m}{k},$$

the preceding relation can be written in the form

$$P^{[\alpha]}_{m+1}(f;x)-P^{[\alpha]}_m(f;x)=$$

$$\sum_{k=1}^{m}\binom{m}{k}\frac{\prod_{r=0}^{k-1}(x+r\alpha)\prod_{s=0}^{m-k}(1-x+s\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+m\alpha)}\left\{\frac{m+1}{m+1-k}f\left(\frac{k}{m+1}\right)-\right.$$

$$\left.-\frac{k}{m+1-k}f\left(\frac{k-1}{m}\right)-f\left(\frac{k}{m}\right)\right\}.$$

If we use the change $k=\nu+1$ and we notice that $\binom{m}{\nu+1}=$ $=\frac{m}{\nu+1}\binom{m-1}{\nu}$, we obtain

$$P^{[\alpha]}_{m+1}(f;x)-P^{[\alpha]}_m(f;x)=$$

$$-m\sum_{\nu=0}^{m-1}\binom{m-1}{\nu}\frac{\prod_{r=0}^{\nu}(x+r\alpha)\prod_{s=0}^{m-\nu-1}(1-x+s\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+m\alpha)}\left\{\frac{1}{m-\nu}f\left(\frac{\nu}{m}\right)-\right.$$

$$\left.-\frac{m+1}{(m-\nu)(\nu+1)}f\left(\frac{\nu+1}{m+1}\right)+\frac{1}{\nu+1}f\left(\frac{\nu+1}{m}\right)\right\}.$$

From the fact that the divided difference of the function f on the nodes $\frac{\nu}{m},\frac{\nu+1}{m+1},\frac{\nu+1}{m}$ is expressible by the formula

$$\left[\frac{\nu}{m},\frac{\nu+1}{m+1},\frac{\nu+1}{m};f\right]=$$

$$=m^2(m+1)\left\{\frac{1}{m-\nu}f\left(\frac{\nu}{m}\right)-\frac{m+1}{(m-\nu)(\nu+1)}f\left(\frac{\nu+1}{m+1}\right)+\frac{1}{\nu+1}f\left(\frac{\nu+1}{m}\right)\right\}$$

and that

$$\frac{x(x+\alpha)\dots(x+\nu\alpha)(1-x)(1-x+\alpha)\dots(1-x+\overline{m-\nu-1}\,\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+m\alpha)}=$$

$$=\frac{(x+\nu\alpha)(1-x+\overline{m-\nu-1}\,\alpha)}{(1+m\alpha)(1+\overline{m-1}\,\alpha)}w_{m-1,\nu}(x;\alpha),$$

there follows the desired formula (2.1).

In particular, when $\alpha=0$ it reduces to the relation corresponding to the Bernstein polynomials (see [1], [18])

$$B_{m+1}(f;x)-B_m(f;x)=-\frac{x(1-x)}{m(m+1)}\sum_{\nu=0}^{m-1}p_{m-1,\nu}(x)\left[\frac{\nu}{m},\frac{\nu+1}{m+1},\frac{\nu+1}{m};f\right].$$

On the other hand, if we set $x=\frac{ny}{m}$ we obtain

$$mx(1-x)=ny\left(1-\frac{ny}{m}\right),$$

$$p_{m-1,\nu}\left(\frac{ny}{m}\right)=\frac{1}{\nu!}\left(1-\frac{1}{m}\right)\dots\left(1-\frac{\nu}{m}\right)(ny)^\nu\left(1-\frac{ny}{m}\right)^{m-1-\nu}$$

and for $m\to\infty$ we get ny, respectively $\frac{(ny)^\nu}{\nu!}e^{-ny}$.

Next, from

$$\frac{1}{m-\nu}f\left(\frac{\nu}{m}\right)-\frac{m+1}{(m-\nu)(\nu+1)}f\left(\frac{\nu+1}{m+1}\right)+\frac{1}{\nu+1}f\left(\frac{\nu+1}{m}\right)=$$

$$=\frac{\frac{1}{m}}{1-\frac{\nu}{m}}f\left(\frac{\nu}{m}\right)-\frac{\frac{1}{m}}{\left(1-\frac{\nu}{m}\right)\frac{\nu+1}{m+1}}f\left(\frac{\nu+1}{m+1}\right)+\frac{\frac{1}{m}}{\frac{\nu+1}{m}}f\left(\frac{\nu+1}{m}\right),$$

by the change of variable $x=\frac{ny}{m}$ we obtain

$$\frac{1}{n}\left\{\frac{1}{1-\frac{\nu}{n}}f\left(\frac{\nu}{n}\right)-\frac{1}{\left(1-\frac{\nu}{n}\right)\frac{\nu+1}{n+1}}f\left(\frac{\nu+1}{n+1}\right)+\frac{1}{\frac{\nu+1}{n}}f\left(\frac{\nu+1}{n}\right)\right\}$$

$$=\frac{1}{n^2(n+1)}\left[\frac{\nu}{n},\frac{\nu+1}{n+1},\frac{\nu+1}{n};f\right].$$

Hence, for the Mirakyan operator (1.4) we have

$$M_{n+1}(f;x)-M_n(f;x)=-\tag{2.2}$$

$$-\frac{x}{n(n+1)}\sum_{\nu=0}^{\infty}\frac{(nx)^\nu}{\nu!}e^{-nx}\left[\frac{\nu}{n},\frac{\nu+1}{n+1},\frac{\nu+1}{n};f\right].$$

5. Since on the interval $[0,1]$ we have

$$(x+\nu\alpha)(1-x+\overline{m-\nu-1}\,\alpha)\,w_{m-1,\nu}(x;\alpha)\geqq 0$$

for $\alpha\geqq 0$ and $\nu=\overline{0,m-1}$, from Theorem 2.1 there follows

COROLLARY 2.1. *If the function f is convex of first order on the interval $[0,1]$*), then the sequence of polynomials $\{P^{[\alpha]}_m(f;x)\}$, where $\alpha\geqq 0$ has a fixed value, is decreasing on the interval (0.1), i.e., $P^{[\alpha]}_m(f;x)>P^{[\alpha]}_{m+1}(f;x)$ for $x\in(0,1)$ and $m=1,2,\dots$.*

*) That is, all its divided differences on three distinct points of [0, 1] are positive.

If f is non-concave of first order on $[0, 1]$*), *then the sequence* $\{P_m^{[\alpha]}(f; x)\}$, *where* $\alpha \geq 0$ *is fixed, is non-increasing on* $[0, 1]$, *i.e.*, $P_m^{[\alpha]}(f; x) \geq P_{m+1}^{[\alpha]}(f; x)$ *for* $x \in [0, 1]$ *and* $m = 1, 2, \ldots$.

By setting $\alpha = 0$ we are led to the monotonicity properties of the sequence $\{B_m(f; x)\}$, studied previously in [20] and [1].

It should be noted that formula (2.2), determined as a limiting case of (2.1), may be used for stating the monotonicity properties of the sequence of Mirakyan's operators, which have been investigated directly, first, by Cheney-Sharma [5].

§ 3. REPRESENTATION OF THE OPERATOR $P_m^{[\alpha]}(f; x)$ BY FINITE AND DIVIDED DIFFERENCES

6. In order to find a representation of the operator (1.1) by means of finite differences we need to establish first

LEMMA 3.1. *Let α be a positive parameter which may depend upon the natural number m. If $x \in (0, 1)$ then*

$$w_{m,k}(x; \alpha) = \binom{m}{k} \frac{B\left(\frac{x}{\alpha} + k, \frac{1-x}{\alpha} + m - k\right)}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)}, \tag{3.1}$$

where

$$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt \qquad (a,\ b > 0)$$

is the beta function.

If $x = 0$ then we have

$$w_{m,k}(0; \alpha) = \begin{cases} 1 \text{ for } k = 0 \\ 0 \text{ for } k = \overline{1, m}, \end{cases} \tag{3.2}$$

and if $x=1$ then

$$w_{m,k}(1; \alpha) = \begin{cases} 0 \text{ for } k = \overline{0, m-1} \\ 1 \text{ for } k = m \end{cases} \tag{3.3}$$

Proof. Since

$$B(a, b) = \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(a+b)},$$

where $\Gamma(c)$ is the gamma function

$$\Gamma(c) = \int_0^\infty x^{c-1} e^{-x}\,dx \quad (c > 0),$$

*) That is, all its divided differences on three distinct points of $[0, 1]$ are non-negative.

and

$$\Gamma(c+n) = c(c+1)\ldots(c+n-1)\,\Gamma(c),$$

if n is a natural number, then we can write

$$B\left(\frac{x}{\alpha} + k, \frac{1-x}{\alpha} + m - k\right) = \frac{\Gamma\left(\frac{x}{\alpha} + k\right)\Gamma\left(\frac{1-x}{\alpha} + m - k\right)}{\Gamma\left(\frac{1}{\alpha} + m\right)} =$$

$$= \frac{\frac{x}{\alpha}\left(\frac{x}{\alpha}+1\right)\ldots\left(\frac{x}{\alpha}+k-1\right)\Gamma\left(\frac{x}{\alpha}\right)\frac{1-x}{\alpha}\left(\frac{1-x}{\alpha}+1\right)\ldots\left(\frac{1-x}{\alpha}+m-k-1\right)\Gamma\left(\frac{1-x}{\alpha}\right)}{\frac{1}{\alpha}\left(\frac{1}{\alpha}+1\right)\ldots\left(\frac{1}{\alpha}+m-1\right)\Gamma\left(\frac{1}{\alpha}\right)} =$$

$$= \frac{w_{m,k}(x; \alpha)}{\binom{m}{k}} B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)$$

Thus the equality (3.1) is proved.

Formulas (3.2) and (3.3) follow directly from (1.2).

7. Using the preceding lemma we may now prove

THEOREM 3.1. *The operator $P_m^{[\alpha]}(f; x)$ can be represented by the following expansion*

$$P_m^{[\alpha]}(f; x) = f(0) + \sum_{j=1}^{m} \binom{m}{j} \frac{x(x+\alpha)\ldots(x+\overline{j-1}\,\alpha)}{(1+\alpha)(1+2\alpha)\ldots(1+\overline{j-1}\,\alpha)} \Delta_{\frac{1}{m}}^{j} f(0), \tag{3.4}$$

where $\Delta_{\frac{1}{m}}^{j} f(0)$ is the finite difference of order j, with the step $\frac{1}{m}$ and the starting point 0 of the function f, that is

$$\Delta_{\frac{1}{m}}^{j} f(0) = \sum_{\nu=0}^{j} (-1)^\nu \binom{j}{\nu} f\left(\frac{j-\nu}{m}\right).$$

Proof. Assuming that $\alpha > 0$, $x \neq 0$ and $x \neq 1$, by use of Lemma 3.1 we can write successively

$$P_m^{[\alpha]}(f; x) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \sum_{k=0}^{m} \binom{m}{k} B\left(\frac{x}{\alpha} + k, \frac{1-x}{\alpha} + m - k\right) f\left(\frac{k}{m}\right) =$$

$$= \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \sum_{k=0}^{m} \binom{m}{k} \left[\int_0^1 t^{\frac{x}{\alpha}+k-1}(1-t)^{\frac{1-x}{\alpha}+m-k-1}\,dt\right] f\left(\frac{k}{m}\right) =$$

$$= \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \sum_{k=0}^{m} \binom{m}{k} \int_0^1 \left[t^{\frac{x}{\alpha}-1} (1-t)^{\frac{1-x}{\alpha}-1} \right] \left[t^k (1-t)^{m-k} f\left(\frac{k}{m}\right)\right] dt =$$

$$= \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \int_0^1 t^{\frac{x}{\alpha}-1} (1-t)^{\frac{1-x}{\alpha}-1} \left[\sum_{k=0}^{m} \binom{m}{k} t^k (1-t)^{m-k} f\left(\frac{k}{m}\right)\right] dt.$$

But according to a result obtained in 1935 by Grüss [8] we have

$$B_m(f;t) = \sum_{k=0}^{m} \binom{m}{k} t^k (1-t)^{m-k} f\left(\frac{k}{m}\right) = \sum_{j=0}^{m} \binom{m}{j} t^j \Delta^j_{\frac{1}{m}} f(0). \tag{3.5}$$

Consequently

$$P_m^{[\alpha]}(f;x) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \sum_{j=0}^{m} \binom{m}{j} \Delta^j_{\frac{1}{m}} f(0) \int_0^1 t^{\frac{x}{\alpha}+j-1} (1-t)^{\frac{1-x}{\alpha}-1} dt = \tag{3.6}$$

$$= \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \sum_{j=0}^{m} \binom{m}{j} \cdot \Delta^j_{\frac{1}{m}} f(0) \cdot B\left(\frac{x}{\alpha}+j, \frac{1-x}{\alpha}\right).$$

Applying repeatedly the following formula

$$B(a+1, b) = \frac{a}{a+b} B(a, b),$$

we obtain the relationship

$$B\left(\frac{x}{\alpha}+j, \frac{1-x}{\alpha}\right) = \frac{x(x+\alpha)\dots(x+\overline{j-1}\,\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+\overline{j-1}\,\alpha)} B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right).$$

By making use of it, the equality (3.6) leads us just to the formula (3.4).

One observes that this formula is true also when $\alpha = 0$ since it is readily seen that in this special case it reduces to Grüss's formula (3.5).

When $x = 0$ and $x = 1$ formula (3.4) is also true because if we replace these values in (1.1) and take (1.2) into account we get

$$P_m^{[\alpha]}(f;0) = f(0), \quad P_m^{[\alpha]}(f;1) = f(1), \tag{3.7}$$

that is, *the polynomial* (1.1) *is interpolating at the ends of the interval* $[0, 1]$.

On the other hand, if we set $x = 0$ and $x = 1$, in (3.4), then we find respectively

$$P_m^{[\alpha]}(f;0) = f(0), \quad P_m^{[\alpha]}(f;1) = \sum_{j=0}^{m} \binom{m}{j} \Delta^j_{\frac{1}{m}} f(0).$$

But the well-known formula from the theory of finite differences

$$f(a+ph) = \sum_{j=0}^{p} \binom{p}{j} \Delta^j_h f(a),$$

for $a = 0$, $p = m$, $h = \frac{1}{m}$ yields

$$f(1) = \sum_{j=0}^{m} \binom{m}{j} \Delta^j_{\frac{1}{m}} f(0) = P_m^{[\alpha]}(f;1).$$

8. From Theorem 3.1 we may derive.

COROLLARY 3.1. *The operator* (1.1) *can be expressed in terms of divided differences as follows*

$$P_m^{[\alpha]}(f;x) = f(0) + \sum_{j=0}^{m} A_{m,j}(f)\, x(x+\alpha)\dots(x+\overline{j-1}\,\alpha), \tag{3.8}$$

where

$$A_{m,j}(f) = \left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\dots\left(1-\frac{j-1}{m}\right)\left[0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{j}{m}; f\right].$$

Proof. By making use of the following relation between divided differences and finite differences

$$[a, a+h, \dots, a+mh; f] = \frac{\Delta^m_h f(a)}{m!\, h^m}$$

we obtain

$$\Delta^j_{\frac{1}{m}} f(0) = \frac{j!}{m^j}\left[0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{j}{m}; f\right]. \tag{3.9}$$

Replacing it in (3.4) we are led to the formula (3.8).

Remark. Since by making the change of variable $x = \frac{ny}{m}$ we obtain $\Delta^j_{\frac{1}{n}} f(0)$ in place of $\Delta^j_{\frac{1}{m}} f(0)$, if we set $\alpha = \frac{1}{m^2}$ then we have

$$\binom{m}{j} \frac{x(x+\alpha)\dots(x+\overline{j-1}\,\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+\overline{j-1}\,\alpha)} =$$

$$= \frac{1}{j!} \frac{\left(1-\frac{1}{m}\right)\dots\left(1-\frac{j-1}{m}\right)}{\left(1+\frac{1}{m^2}\right)\dots\left(1+\frac{j-1}{m^2}\right)} ny\left(ny+\frac{1}{m}\right)\dots\left(ny+\frac{j-1}{m}\right)$$

and this tends to $\frac{(ny)^j}{j!}$ as m tends to ∞.

Denoting again the variable by x, we see that from (3.4) we can deduce for the Mirakyan operator (1.4) the following representation

$$M_n(f;x) = \sum_{j=0}^{\infty} \frac{(nx)^j}{j!} \Delta^j_{\frac{1}{n}} f(0).$$

According to the relation (3.9) it follows that we also can write the formula

$$M_n(f;x)=\sum_{j=0}^{\infty}\left[0,\frac{1}{n},\ldots,\frac{j}{n};f\right]x^j,$$

which was established directly in [12].

§ 4. CONVERGENCE PROPERTY OF THE SEQUENCE $\{P_m^{[\alpha]}(f;x)\}$

9. In this section we proceed to the topic of convergence of the sequence of polynomials (1.1).

We need first to establish

LEMMA 4.1. *The following identities*

$$P_m^{[\alpha]}(1;x)=1,\quad P_m^{[\alpha]}(t;x)=x,\quad P_m^{[\alpha]}(t^2;x)= \tag{4.1}$$
$$=\frac{1}{1+\alpha}\left[\frac{x(1-x)}{m}+x(x+\alpha)\right]$$

hold.

Proof. We first note that

$$\Delta_h^r x^n=\begin{cases}0 \text{ if } r>n\\ r!\,h^r \text{ if } r=n.\end{cases}$$

By setting $f(t)=1$ in (3.4) we obtain immediately the first relation from (4.1). Taking (1.1) into consideration it may be written under the form

$$\sum_{k=0}^{m} w_{m,k}(x;\alpha)=1. \tag{4.2}$$

If we set $f(t)=t$ and $f(t)=t^2$ in (3.4) then because of the fact that

$$\Delta_{\frac{1}{m}}t\Big|_{t=0}=\frac{1}{m},\quad \Delta_{\frac{1}{m}}t^2\Big|_{t=0}=\frac{1}{m^2},\quad \Delta_{\frac{1}{m}}^2 t^2\Big|_{t=0}=\frac{2}{m^2},$$

we obtain

$$P_m^{[\alpha]}(t;x)=\binom{m}{1}x\cdot\frac{1}{m}=x,$$

$$P_m^{[\alpha]}(t^2;x)=\binom{m}{1}x\cdot\frac{1}{m^2}+\binom{m}{2}\frac{x(x+\alpha)}{1+\alpha}\cdot\frac{2}{m^2}=\frac{1}{1+\alpha}\left[\frac{x(1-x)}{m}+x(x+\alpha)\right].$$

10. Now we state and prove the central result of this paper.

THEOREM 4.1. *If $f\in C[0,1]$ and $0\leqq\alpha=\alpha(m)\to 0$ as $m\to\infty$, then the sequence $\{P_m^{[\alpha]}(f;x)\}$ converges to $f(x)$ uniformly on $[0,1]$.*

Proof. By making use of the identities (4.1) we can write

$$\lim_{m\to\infty}P_m^{[\alpha]}(t^r;x)=x^r\ (r=0,1,2)$$

uniformly on $[0,1]$. Consequently, our assertion appears directly from the well-known theorem of Bohman-Korovkin.

§ 5. ESTIMATE OF THE ORDER OF APPROXIMATION

11. In this section we are concerned with the estimate of the order of approximation of a function $f\in C[0,1]$ by our operator (1.1).

It may be simply described by means of the modulus of continuity, defined by

$$\omega(\delta)=\omega(f;\delta)=\sup|f(x'')-f(x')|,$$

where x' and x'' are points from $[0,1]$ so that $|x''-x'|<\delta$, δ being a positive number.

We may now prove

THEOREM 5.1. *If $f\in C[0,1]$ and $\alpha\geqq 0$, then*

$$|f(x)-P_m^{[\alpha]}(f;x)|\leqq\frac{3}{2}\,\omega\left(\sqrt{\frac{1+\alpha m}{m+\alpha m}}\right). \tag{5.1}$$

Proof. By the identity (4.2) and the fact that $w_{m,k}(x;\alpha)\geqq 0$, we can write

$$|f(x)-P_m^{[\alpha]}(f;x)|=\left|\sum_{k=0}^{m}w_{m,k}(x;\alpha)\left[f(x)-f\left(\frac{k}{m}\right)\right]\right|\leqq$$
$$\leqq\sum_{k=0}^{m}w_{m,k}(x;\alpha)\left|f(x)-f\left(\frac{k}{m}\right)\right|.$$

Using the following known properties of the modulus of continuity

$$|f(x'')-f(x')|\leqq\omega(|x''-x'|),$$
$$\omega(\lambda\delta)\leqq([\lambda]+1)\,\omega(\delta)\leqq(\lambda+1)\,\omega(\delta)\qquad(\lambda>0) \tag{5.2}$$

we obtain

$$\left|f(x)-f\left(\frac{k}{m}\right)\right|\leqq\omega\left(\left|x-\frac{k}{m}\right|\right)=\omega\left(\frac{1}{\delta}\left|x-\frac{k}{m}\right|\delta\right)\leqq$$
$$\leqq\left(1+\frac{1}{\delta}\left|x-\frac{k}{m}\right|\right)\omega(\delta).$$

Consequently

$$|f(x)-P_m^{[\alpha]}(f;x)|\leqq\left(1+\frac{1}{\delta}\sum_{k=0}^{m}w_{m,k}(x;\alpha)\left|x-\frac{k}{m}\right|\right)\omega(\delta). \tag{5.3}$$

But according to Cauchy's inequality we have

$$\sum_{k=0}^{m} w_{m,k}(x;\alpha)\left|x-\frac{k}{m}\right| \leq \left[\sum_{k=0}^{m} w_{m,k}(x;\alpha)\left(x-\frac{k}{m}\right)^2\right]^{\frac{1}{2}}.$$

Then by virtue of the linearity of the operator (1.1) and by Lemma 4.1 we obtain

$$\sum_{k=0}^{m} w_{m,k}(x;\alpha)\left(x-\frac{k}{m}\right)^2 = x^2 P_m^{[\alpha]}(1;x) - 2x P_m^{[\alpha]}(t;x) + P_m^{[\alpha]}(t^2;x)$$

$$(5.4)\qquad = x^2 - 2x^2 + \frac{1}{1+\alpha}\left[\frac{x(1-x)}{m} + x(x+\alpha)\right] = \frac{1+\alpha m}{1+\alpha}\cdot\frac{x(1-x)}{m} \leq$$

$$\leq \frac{1+\alpha m}{4(m+\alpha m)},$$

and therefore

$$(5.5)\qquad \sum_{k=0}^{m} w_{m,k}(x;\alpha)\left|x-\frac{k}{m}\right| \leq \frac{1}{2}\sqrt{\frac{1+\alpha m}{m+\alpha m}}.$$

The inequalities (5.3) and (5.5) enables us to find

$$|f(x) - P_m^{[\alpha]}(f;x)| \leq \left(1+\frac{1}{2\delta}\sqrt{\frac{1+\alpha m}{m+\alpha m}}\right)\omega(\delta).$$

By inserting into it

$$\delta = \sqrt{\frac{1+\alpha m}{m+\alpha m}},$$

we are led just to the desired inequality (5.1).

Assuming that $\alpha = \alpha(m) \to 0$ as $m \to \infty$ we see that Theorem 4.1 is an immediate consequence of Theorem 5.1.

One observes that for $\alpha = 0$ the inequality (5.1) reduces to the known inequality of Popoviciu [16] corresponding to the Bernstein polynomial (1.3):

$$|f(x) - B_m(f;x)| \leq \frac{3}{2}\,\omega\left(\frac{1}{\sqrt{m}}\right).$$

If we set $\alpha = m^{-1}$ and take into consideration the second property from (5.2) of the modulus of continuity, we obtain

$$|f(x) - P_m^{\left[\frac{1}{m}\right]}(f;x)| \leq 3\,\omega\left(\frac{1}{\sqrt{m+1}}\right).$$

Remark. One sees that in the case of Mirakyan's operator (1.4) instead of the relation (5.4) we have

$$e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\left(x-\frac{k}{m}\right)^2 = x^2 M_n(1;x) - 2x M_n(t;x) + M_n(t^2;x) = \frac{x}{n} \leq \frac{a}{n}$$

if $x \in [0, a]$ and therefore the choice $\delta = \frac{1}{\sqrt{n}}$ permits us to obtain the inequality

$$|f(x) - M_n(f;x)| \leq (1+\sqrt{a})\,\omega\left(\frac{1}{\sqrt{n}}\right),$$

which was found recently by Müller [14]. Actually, it should be noticed that it can also be obtained from a more general inequality of Baskakov [2].

12. If one further assumes that the function f possesses a continuous derivative on the interval [0, 1], then we may proceed to give a new estimate of the order of approximation of f by means of the operator (1.1).

We have

THEOREM 5.2. *If $f \in C^1$ [0,1], then the following inequality*

$$(5.6)\qquad |f(x) - P_m^{[\alpha]}(f;x)| \leq \frac{3}{4}\sqrt{\frac{1+\alpha m}{m+\alpha m}}\,\omega_1\left(\sqrt{\frac{1+\alpha m}{m+\alpha m}}\right)$$

holds, where $\omega_1(\delta)$ is the modulus of continuity of f'.

Proof. By applying the mean value theorem of differential calculus we can write

$$f(x) - f\left(\frac{k}{m}\right) = \left(x-\frac{k}{m}\right)f'(\xi) = \left(x-\frac{k}{m}\right)f'(x) +$$

$$+\left(x-\frac{k}{m}\right)[f'(\xi) - f'(x)],$$

where $\xi = \xi_{m,k}(x)$ is an interior point of the interval determined by x and $\frac{k}{m}$.

If we multiply both members of this equality by $w_{m,k}(x;\alpha)$ and sum over k, there follows

$$f(x) - P_m^{[\alpha]}(f;x) = f'(x)\sum_{k=0}^{m}\left(x-\frac{k}{m}\right)w_{m,k}(x;\alpha)$$

$$+\sum_{k=0}^{m}\left(x-\frac{k}{m}\right)w_{m,k}(x;\alpha)\,[f'(\xi) - f'(x)].$$

But according to the relations (4.1) we observe that the first sum vanishes and we thus obtain

$$|f(x) - P_m^{[\alpha]}(f;x)| \leq \sum_{k=0}^{m}\left|x-\frac{k}{m}\right| w_{m,k}(x;\alpha)\,|f'(\xi) - f'(x)|.$$

Then in accordance with (5.2) we have

$$|f'(\xi) - f'(x)| \leq \omega_1(|\xi - x|) \leq \left(1 + \frac{1}{\delta}|\xi - x|\right)\omega_1(\delta) \leq$$

$$\leq \left(1 + \frac{1}{\delta}\left|\frac{k}{m} - x\right|\right)\omega_1(\delta),$$

where δ is a positive number which does not depend on k. Consequently, we can write

$$|f(x) - P_m^{[\alpha]}(f;x)| \leq \left\{\sum_{k=0}^{m}\left|x - \frac{k}{m}\right| w_{m,k}(x;\alpha) + \right.$$

$$\left. + \frac{1}{\delta}\sum_{k=0}^{m}\left(x - \frac{k}{m}\right)^2 w_{m,k}(x;\alpha)\right\}\omega_1(\delta).$$

Further, by using the relationships (5.4) and (5.5) we deduce that

$$|f(x) - P_m^{[\alpha]}(f;x)| \leq \frac{1}{2}\sqrt{\frac{1+\alpha m}{m+\alpha m}}\left(1 + \frac{1}{2\delta}\sqrt{\frac{1+\alpha m}{m+\alpha m}}\right)\omega_1(\delta).$$

By choosing

$$\delta = \sqrt{\frac{1+\alpha m}{m+\alpha m}}$$

we are led to the inequality (5.6) which we desired to establish.

In the special case when $\alpha = 0$ it reduces to the inequality

$$|f(x) - B_m(f;x)| \leq \frac{3}{4}\frac{1}{\sqrt{m}}\,\omega_1\left(\frac{1}{\sqrt{m}}\right),$$

due to Lorentz [10].

It is readily seen that in the case of the operator (1.4) of Mirakyan we have

$$|f(x) - M_n(f;x)| \leq (a + \sqrt{a})\frac{1}{\sqrt{n}}\,\omega_1\left(\frac{1}{\sqrt{n}}\right),$$

where a is a positive number and $x \in [0, a]$.

§ 6. EVALUATION OF THE REMAINDER

13. We shall deal now with the evaluation of the remainder term $R_m^{[\alpha]}(f;x)$ of the approximation formula

(6.1) $$f(x) = P_m^{[\alpha]}(f;x) + R_m^{[\alpha]}(f;x)$$

of the function f by the operator (1.1).

It is quite easy to prove

THEOREM 6.1. *The remainder $R_m^{[\alpha]}(f;x)$ of the approximation formula (6.1) can be represented in the form*

$$R_m^{[\alpha]}(f;x) = \frac{x-1}{m(1+\overline{m-1}\,\alpha)}\sum_{k=0}^{m-1}(x+\alpha k)\,w_{m-1,k}(x;\alpha)\left[x, \frac{k}{m}, \frac{k+1}{m}; f\right] +$$

(6.2)

$$+ \frac{\alpha}{1+\overline{m-1}\,\alpha}\sum_{k=0}^{m-1}(\overline{m-1}\,x - k)\,w_{m-1,k}(x;\alpha)\left[x, \frac{k}{m}; f\right].$$

Proof. For the sake of brevity, let us first introduce the notation

$$v_{m,k}(x;\alpha) = \frac{x(x+\alpha)\ldots(x+\overline{k-1}\,\alpha)(1-x)(1-x+\alpha)\ldots(1-x+\overline{m-k-1}\,\alpha)}{(1+\alpha)(1+2\alpha)\ldots(1+\overline{m-1}\,\alpha)}$$

Thus we have: $w_{m,k}(x;\alpha) = \binom{m}{k} v_{m,k}(x;\alpha)$.

One observes that we can write successively

$$R_m^{[\alpha]}(f;x) = f(x) - P_m^{[\alpha]}(f;x) = \sum_{k=0}^{m} w_{m,k}(x;\alpha)\left[f(x) - f\left(\frac{k}{m}\right)\right] =$$

$$= \sum_{k=0}^{m} w_{m,k}(x;\alpha)\left(x - \frac{k}{m}\right)\left[x, \frac{k}{m}; f\right] =$$

$$= \frac{1}{m}\sum_{k=0}^{m}\binom{m}{k} v_{m,k}(x;\alpha)(mx - k)\left[x, \frac{k}{m}; f\right].$$

If we use the identity: $mx - k = (m-k)x - (1-x)k$, then we obtain

$$R_m^{[\alpha]}(f;x) = \frac{x}{m}\sum_{k=0}^{m-1}(m-k)\binom{m}{k} v_{m,k}(x;\alpha)\left[x, \frac{k}{m}; f\right]$$

$$- \frac{1-x}{m}\sum_{k=1}^{m} k\binom{m-1}{k} v_{m,k}(x;\alpha)\left[x, \frac{k}{m}; f\right].$$

Since

$$(m-k)\binom{m}{k} = m\binom{m-1}{k},\quad k\binom{m}{k} = m\binom{m-1}{k-1},$$

there follows

$$R_m^{[\alpha]}(f;x) = x\sum_{k=0}^{m-1}\binom{m-1}{k} v_{m,k}(x;\alpha)\left[x, \frac{k}{m}; f\right] -$$

$$-(1-x)\sum_{k=1}^{m}\binom{m-1}{k-1} v_{m,k}(x;\alpha)\left[x, \frac{k}{m}; f\right].$$

Effecting in the last sum the change $k=i+1$ and then denoting again the summation variable by k we get

$$R_m^{[\alpha]}(f;x) = x\sum_{k=0}^{m-1}\binom{m-1}{k} v_{m,k}(x;\alpha)\left[x, \frac{k}{m}; f\right] -$$

$$-(1-x)\sum_{k=0}^{m-1}\binom{m-1}{k} v_{m,k+1}(x;\alpha)\left[x, \frac{k+1}{m}; f\right].$$

Since

$$v_{m,k}(x;\alpha) = \frac{1-x+\overline{m-k-1}\,\alpha}{1+\overline{m-1}\,\alpha}\, v_{m-1,k}(x;\alpha),$$

$$v_{m,k+1}(x;\alpha) = \frac{x+k\alpha}{1+\overline{m-1}\,\alpha}\, v_{m-1,k}(x;\alpha),$$

it follows that

$$R_m^{[\alpha]}(f;x) = \frac{1}{1+\overline{m-1}\,\alpha}\sum_{k=0}^{m-1} w_{m-1,k}(x;\alpha)\Big\{x(1-x+\overline{m-k-1}\,\alpha)\times$$

$$\times\left[x, \frac{k}{m}; f\right] - (1-x)(x+\alpha k)\left[x, \frac{k+1}{m}; f\right]\Big\}.$$

Because of the fact that

$$x(1-x+\overline{m-k-1}\,\alpha) = (1-x)(x+\alpha k) + \alpha(\overline{m-1}\,x-k),$$

we obtain

$$R_m^{[\alpha]}(f;x) = \frac{1-x}{1+\overline{m-1}\,\alpha}\sum_{k=0}^{m-1} w_{m-1,k}(x;\alpha)(x+\alpha k)\left(\left[x, \frac{k}{m}; f\right] -\right.$$

$$\left.-\left[x, \frac{k+1}{m}; f\right]\right) +$$

$$+\frac{\alpha}{1+\overline{m-1}\,\alpha}\sum_{k=0}^{m-1}(\overline{m-1}\,x-k)\, w_{m-1,k}(x;\alpha)\left[x, \frac{k}{m}; f\right].$$

Finally, the use of the recurrence formula of divided differences gives

$$\left[x, \frac{k}{m}; f\right] - \left[x, \frac{k+1}{m}; f\right] = -\frac{1}{m}\left[x, \frac{k}{m}, \frac{k+1}{m}; f\right]$$

and therefore we see that the remainder $R_m^{[\alpha]}(f;x)$ is expressible in the form (6.2).

One observes that when $\alpha = 0$ it reduces to

$$R_m^{[0]}(f;x) = R_m(f;x) = -\frac{x(1-x)}{m}\sum_{k=0}^{m-1} p_{m-1,k}(x)\left[x, \frac{k}{m}, \frac{k+1}{m}; f\right],$$

which corresponds to the approximation of f by the Bernstein polynomial (1.3). This was first established by us in [17].

Taking $x = \frac{ny}{m}$ we see that for $m \to \infty$ the preceding relation leads us to the following expression for the remainder of approximation formula by the Mirakyan operator

$$R_n(f;x) = -\frac{x}{n}\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\left[x, \frac{k}{n}, \frac{k+1}{n}; f\right].$$

We will also consider here the special case when $\alpha = \frac{1}{m-1}$, which yields

$$R_m^{\left[\frac{1}{m-1}\right]}(f;x) = \frac{x-1}{2m}\sum_{k=0}^{m-1}\left(x+\frac{k}{m-1}\right) w_{m-1,k}\left(x;\frac{1}{m-1}\right)\left[x, \frac{k}{m}, \frac{k+1}{m}; f\right] +$$

$$+\frac{1}{2}\sum_{k=0}^{m-1}\left(x-\frac{k}{m-1}\right) w_{m-1,k}\left(x;\frac{1}{m-1}\right)\left[x, \frac{k}{m}; f\right].$$

14. From Theorem 6.1 we may deduce

COROLLARY 6.1. *If on the interval* $[0, 1]$ *the divided differences of first and second order of the function* f *are bounded, then we have*

$$(6.3)\quad |R_m^{[\alpha]}(f;x)| \leqq \frac{x(1-x)}{m} M_2(f) + \frac{1}{2}\sqrt{\frac{\alpha}{1+\alpha}}\sqrt{\frac{\overline{m-1}\,\alpha}{1+\overline{m-1}\,\alpha}}\, M_1(f),$$

where $M_1(f)$ *and* $M_2(f)$ *are the least upper bounds of the absolute values of first and respectively of second order divided differences of* f *on* $[0, 1]$.

Proof. From (6.2) we deduce immediately

$$(6.4)\quad |R_m^{[\alpha]}(f;x)| \leqq \frac{1-x}{m(1+\overline{m-1}\,\alpha)} M_2(f)\sum_{k=0}^{m-1}(x+\alpha k)\, w_{m-1,k}(x;\alpha)$$

$$+\frac{(m-1)\alpha}{1+\overline{m-1}\,\alpha} M_1(f)\sum_{k=0}^{m-1} w_{m-1,k}(x;\alpha)\left|x-\frac{k}{m-1}\right|.$$

But according to the relationships (4.1) and (5.5) we have

$$\sum_{k=0}^{m-1}(x+\alpha k)\, w_{m-1,k}(x;\alpha) = x(1+\overline{m-1}\,\alpha),$$

$$\sum_{k=0}^{m-1} w_{m-1,k}(x;\alpha)\left|x-\frac{k}{m-1}\right| \leqq \frac{1}{2}\sqrt{\frac{1+(m-1)\alpha}{(m-1)(1+\alpha)}}.$$

Consequently, we arrive at the desired inequality.

Remark. Assuming that f has a finite second order derivative on $[0, 1]$ and denoting by $N_1(f)$ and $N_2(f)$ the least upper bounds of $|f'|$ and respectively of $|f''|$ on $[0, 1]$, then in the inequality (6.3) we may replace $M_1(f)$ by $N_1(f)$ and $M_2(f)$ by $\frac{1}{2}N_2(f)$.

§ 7. ASYMPTOTIC ESTIMATE OF THE REMAINDER

15. We shall end this paper by extending to our operator (1.1) an asymptotic estimate of remainder given by Voronovskaja (see, e.g., [10]) in the case of Bernstein's polynomials.

We have

THEOREM 7.1. *Let $\alpha = \alpha(m) \to 0$ as $m \to \infty$. If f is bounded on $[0, 1]$ and possesses a second derivative at a point x of $[0, 1]$, then*

$$R_m^{[\alpha]}(f; x) = f(x) - P_m^{[\alpha]}(f; x) = -\frac{1+\alpha m}{1+\alpha} \cdot \frac{x(1-x)}{2m} f''(x) + \frac{\varepsilon_m^{[\alpha]}(x)}{m}, \tag{7.1}$$

where $\varepsilon_m^{[\alpha]}(x)$ tends to 0 when m tends to ∞.

Proof. Let $y \in [0, 1]$. It is well known that if $f(y)$ has a finite second order derivative at a point $x \in [0, 1]$ then there exists a function $g(y)$, defined on $[0, 1]$, so that for $y \to x$ we have $g(y) \to 0$ and that $f(y)$ can be expanded by Taylor's formula

$$f(y) = f(x) + (y-x) f'(x) + (y-x)^2 \left[\frac{1}{2} f''(x) + g(y)\right].$$

If we replace here $y = \frac{k}{m}$, multiply both members by $w_{m,k}(x; \alpha)$ and then sum over k, we obtain

$$P_m^{[\alpha]}(f; x) = f(x) + f'(x) \sum_{k=0}^{m} w_{m,k}(x; \alpha)\left(\frac{k}{m} - x\right) + \frac{1}{2} f''(x) \sum_{k=0}^{m} w_{m,k}(x; \alpha)\left(\frac{k}{m} - x\right)^2 + \rho_m^{[\alpha]}(x), \tag{7.2}$$

where

$$\rho_m^{[\alpha]}(x) = \sum_{k=0}^{m} w_{m,k}(x; \alpha)\left(\frac{k}{m} - x\right)^2 g\left(\frac{k}{m}\right). \tag{7.3}$$

By virtue of (4.1) the first sum from the right-hand side of (7.2) vanishes, while the second sum is estimated at (5.4).

Consequently,

$$-R_m^{[\alpha]}(f; x) = \frac{1+\alpha m}{1+\alpha} \cdot \frac{x(1-x)}{2m} f''(x) + \rho_m^{[\alpha]}(x).$$

Because of the fact that $g(y) \to 0$ as $y \to x$, it follows that for every number $\varepsilon > 0$ there exists another number $\delta > 0$ such that $|y - x| \leq \delta$ implies $|g(y)| < \varepsilon$. Taking $y = \frac{k}{m}$ we must have

$$\left| g\left(\frac{k}{m}\right)\right| < \varepsilon \quad \text{if} \quad \left|\frac{k}{m} - x\right| \leq \delta. \tag{7.4}$$

According to (7.3) we have

$$|\rho_m^{[\alpha]}(x)| \leq \sum_{k=0}^{m} \left(\frac{k}{m} - x\right)^2 w_{m,k}(x; \alpha) \left| g\left(\frac{k}{m}\right)\right|.$$

Now let us split the above sum into two parts: the first being extended over the set I_m of integers k which satisfy the inequality $\left|\frac{k}{m} - x\right| \leq \delta$, and the second extended to the set J_m of integers k for which $\left|\frac{k}{m} - x\right| > \delta$. Therefore

$$|\rho_m^{[\alpha]}(x)| \leq \sum_{k \in I_m} \left(\frac{k}{m} - x\right)^2 w_{m,k}(x; \alpha)\left| g\left(\frac{k}{m}\right)\right| + \sum_{k \in J_m} \left(\frac{k}{m} - x\right)^2 w_{m,k}(x; \alpha)\left| g\left(\frac{k}{m}\right)\right|$$

Since for $x, y \in [0, 1]$ the function $(y-x)^2 |g(y)|$ is bounded (say by M), in accordance with (7.4) we can write

$$|\rho_m^{[\alpha]}(x)| < \varepsilon \frac{1+\alpha m}{1+\alpha} \cdot \frac{x(1-x)}{m} + M \sum_{k \in J_m} w_{m,k}(x; \alpha).$$

But $x(1-x) \leq \frac{1}{4}$ and given the positive numbers δ and η, there exists a positive integer $N(\delta, \eta)$ such that $m > N(\delta, \eta)$ implies

$$\sum_{k \in J_m} w_{m,k}(x; \alpha) < \eta.$$

Hence

$$|\rho_m^{[\alpha]}(x)| < \frac{\varepsilon(1+\alpha m)}{4(1+\alpha)m} + M\eta.$$

Choosing

$$\eta = \frac{(3 + 4\alpha - \alpha m)\varepsilon}{4(1+\alpha)mM}$$

we obtain

$$|\rho_m^{[\alpha]}(x)| < \frac{\varepsilon}{m} \quad \text{or} \quad m \cdot |\rho_m^{[\alpha]}(x)| < \varepsilon,$$

whenever m is greater than a certain positive integer $m_0(\varepsilon)$.

Denoting $m \cdot \rho_m^{[\alpha]}(x)$ by $-\varepsilon_m^{[\alpha]}(x)$ we have

$$\rho_m^{[\alpha]}(x) = -\frac{\varepsilon_m^{[\alpha]}(x)}{m},$$

where $\varepsilon_m^{[\alpha]}(x) \to 0$ as $m \to \infty$.

This completes the proof of Theorem 7.1.

Received October 13, 1967

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PROCESSUS DE MARKOV ASSOCIÉS À CERTAINS GROUPES DE LIE

PAR

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L'auteur indique en quelles conditions un processus de Markov continu à n états peut être associé à un groupe simplement transitif G_n.

Un groupe G_2 simplement transitif non abélien est associé à un processus de Markov à deux états dans tout l'espace $E_2(x, y)$.

Aussi les groupes intégrables, ayant la structure (6), admettent un processus de Markov continu dans tout l'espace euclidien si les quantités $\lambda_h (h = 3, \dots, n)$ sont toutes positives.

De même on montre qu'à un groupe nilpotent on ne peut pas associer un processus probabilistique.

On sait qu'un processus de Markov continu à n états est défini par un système de n vecteurs indépendants. Si nous notons ces vecteurs avec $\mu^i_a (i, a = 1, 2, \dots, n)$ et avec λ^a_i les vecteurs duals, alors les probabilités de passage sont données par les formules [1]

$$P^i_j(s, t) = \sum_{a=1}^{n} \mu^i_a(s)\, \lambda^a_j(t), \qquad (i, j = 1, \dots, n, s < t), \tag{1}$$

où $P^i_j(s,t)$ doivent être des quantités positives et les vecteurs λ^a_i doivent satisfaire aux conditions de normalité

$$\sum_{i=1}^{n} \lambda^a_i = 1, \qquad (a = 1, \dots, n). \tag{2}$$

D'autre part étant donné un groupe simple transitif G_n, celui-ci peut être défini comme groupe qui laisse invariante n formes de Pfaff

$$ds^a = \lambda^a_i dx^i, \qquad (i = 1, \dots, n), \tag{3}$$

donc le groupe G_n introduit de même un système de n vecteurs.

REV. ROUM. MATH. PURES ET APPL., TOME XIII, No 8, p. 1195–1200, BUCAREST, 1968