THE RATE OF CONVERGENCE OF POSITIVE LINEAR OPERATORS IN WEIGHTED SPACES

Abstract. We estimate the rate of approximation of positive linear operators for unbounded functions defined on the positive semi-axis, in terms of the modulus of continuity of the first order and the rate of convergence of the function toward infinity.

1. Introduction

Let $\mathbb{R}_+ = [0, \infty)$ and let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be an unbounded strictly increasing continuous function with $\varphi(0) = 0$ and with the property that φ^{-1} is uniformly continuous. Let $\rho(x) = 1 + \varphi^2(x)$ be a weight function and let $B_\rho(\mathbb{R}_+)$ be the Banach space defined by

$$B_\rho(\mathbb{R}_+) = \{\, f : \mathbb{R}_+ \to \mathbb{R} \mid \text{ there exists } M > 0 \text{ such that } |f(x)| \leq M \cdot \rho(x), \text{ for all } x \geq 0 \,\}.$$

This weighted space can be endowed with the ρ-norm

$$\|f\|_\rho = \sup_{x \geq 0} \frac{|f(x)|}{\rho(x)}.$$

We define also the subspaces

$$\begin{aligned} C_\rho(\mathbb{R}_+) &= \{\, f \in B_\rho(\mathbb{R}_+),\ f \text{ is continuous} \,\}, \\ C_\rho^k(\mathbb{R}_+) &= \left\{ f \in C_\rho(\mathbb{R}_+),\ \lim_{x \to +\infty} \frac{f(x)}{\rho(x)} = K_f < +\infty \right\}. \end{aligned}$$

In [1] is given the following Korovkin-type theorem

Theorem 1.1. *If $A_n : C_\rho(\mathbb{R}) \to B_\rho(\mathbb{R})$ is a sequence of positive linear operators such that*

$$\lim_{n \to \infty} \left\| A_n \varphi^i - \varphi^i \right\|_\rho = 0, \quad i = 0, 1, 2,$$

then for any function $f \in C_\rho^k(\mathbb{R})$ we have

$$\lim_{n \to \infty} \|A_n f - f\|_\rho = 0.$$

In [2] it is given an estimation of the rate of convergence for positive linear operators of the following type

$$B_n f(x) = \begin{cases} A_n f(x), & x \leq \eta_n \\ f(x), & x > \eta_n, \end{cases}$$

where $(A_n)_{n \in \mathbb{N}}$ is a sequence of positive linear operators acting from $C_\rho(\mathbb{R}_+))$ to $B_\rho(\mathbb{R}_+)$ and $(\eta_n)_{n \in \mathbb{N}}$ is a sequence converging to infinity when n aproaches infinity. The result is the following

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Theorem 1.2. *Let $A_n : C_\rho(\mathbb{R}_+) \to B_\rho(\mathbb{R}_+)$ be a sequence of positive linear operators with*

$$\begin{aligned} \left\|A_n\varphi^0 - \varphi^0\right\|_{\rho^0} &= a_n, \\ \left\|A_n\varphi - \varphi\right\|_{\rho^{\frac{1}{2}}} &= b_n, \\ \left\|A_n\varphi^2 - \varphi^2\right\|_{\rho} &= c_n, \\ \left\|A_n\varphi^3 - \varphi^3\right\|_{\rho^{\frac{3}{2}}} &= d_n, \end{aligned}$$

where a_n, b_n, c_n and d_n tend to zero as n goes to the infinity. Let η_n be a sequence of real numbers such that

$$\lim_{n\to\infty} \eta_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \rho^{\frac{1}{2}}(\eta_n)\delta_n = 0,$$

where $\delta_n = 2\sqrt{(a_n + 2b_n + c_n)(1 + a_n)} + a_n + 3b_n + 3c_n + d_n$. Then for every $f \in C_\rho(\mathbb{R}_+)$

$$\sup_{0\le x\le \eta_n} \frac{|A_n f(x) - f(x)|}{\rho(x)} \le (7 + 4a_n + 2c_n)\cdot \omega_\varphi\left(f, \rho^{\frac{1}{2}}(\eta_n)\delta_n\right) + \|f\|_\rho\, a_n,$$

where ω_φ is defined for $f \in C_\rho(\mathbb{R}_+)$ and $\delta \ge 0$ by

$$\omega_\varphi(f,\delta) = \sup_{\substack{x,y\ge 0 \\ |\varphi(x)-\varphi(y)|\le\delta}} \frac{|f(x) - f(y)|}{\rho(x) + \rho(y)}.$$

In the same paper it is proved that if φ satisfies also the condition

$$|x - y| \le M|\varphi(x) - \varphi(y)|^\alpha, \text{ for every } x, y \ge 0,$$

where $M > 0$ and $\alpha \in (0, 1]$, then the right-hand side of the estimation from the above theorem tends to 0.

In the present paper, we want to give an estimation of the rate of convergence of $A_n f$ toward f in the general case of any sequence of positive linear operators $A_n : C_\rho(\mathbb{R}_+) \to B_\rho(\mathbb{R}_+)$. For this, we need the following modulus of continuity

$$\omega_\varphi(f,\delta) = \sup_{\substack{x,t\ge 0 \\ |\varphi(x)-\varphi(t)|\le\delta}} |f(x) - f(t)|,$$

defined for every bounded function $f : \mathbb{R}_+ \to \mathbb{R}$ and every $\delta \ge 0$. This modulus is an nonnegative, increasing, bounded function in δ and has the following properties

Proposition 1.1. *For every bounded and uniformly continuous function $f : \mathbb{R}_+ \to \mathbb{R}$ we have*

$$\lim_{n\to\infty} \omega_\varphi(f, \delta_n) = 0 \text{ whenever } \delta_n \to 0.$$

Proof This property is true due to the property of φ^{-1} to be uniformly continuous and due to the following representation

$$\omega_\varphi(f,\delta) = \omega(f \circ \varphi^{-1}, \delta).$$

Indeed, $f\circ\varphi^{-1}$ is a bounded uniformly continuous function and the usual modulus of continuity ω is continuous in 0 for such a function. □

Remark 1.1. *If $\varphi(x) = x$, then ω_φ reduces to the usual modulus of continuity. If φ^{-1} is a Holder function, i.e. there exist $M > 0$ and $\alpha \in (0, 1]$ such that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ has the property*

$$|x - y| \le M|\varphi(x) - \varphi(y)|^\alpha, \text{ for every } x, y \ge 0,$$

then we have the relation

$$\omega_\varphi(f,\delta) \le \omega(f, M\delta^\alpha).$$

Proposition 1.2. *For every bounded function $f : \mathbb{R}_+ \to \mathbb{R}$ and for every $t, x, \delta \geq 0$ we have*

$$|f(t) - f(x)| \leq \left(1 + \frac{[\varphi(t) - \varphi(x)]^2}{\delta^2}\right) \omega_\varphi(f, \delta).$$

Proof We prove first, that for every $m \in \mathbb{N}$

$$\omega_\varphi(f, m\delta) \leq m \cdot \omega_\varphi(f, \delta).$$

For $m = 0$ and $m = 1$ it holds with equality. For $m \geq 2$, let $t > x \geq 0$ such that we have $\varphi(t) - \varphi(x) \leq m\delta$. We construct the points $x_0 = x < x_1 < \cdots < x_m = t$ with the property

$$\varphi(x_k - \varphi(x_{k-1}) = \frac{\varphi(t) - \varphi(x)}{m} \leq \delta.$$

We obtain

$$|f(t) - f(x)| \leq \sum_{k=1}^{m} |f(x_k - f(x_{k-1})| \leq \sum_{k=1}^{m} \omega_\varphi(f, \delta) \leq m \cdot \omega_\varphi(f, \delta).$$

This proves the relation $\omega_\varphi(f, m\delta) \leq m \cdot \omega_\varphi(f, \delta)$. Considering $\lambda > 0$ we obtain

$$\omega_\varphi(f, \lambda\delta) \leq \omega_\varphi(f, ([\lambda] + 1)\delta) \leq ([\lambda] + 1)\omega_\varphi(f, \delta) \leq (\lambda + 1)\omega_\varphi(f, \delta).$$

Now, we can prove the relation from the proposition. Let $t, x \geq 0$. We have

$$|f(t) - f(x)| \leq \omega_\varphi(f, |\varphi(t) - \varphi(x)|).$$

If $|\varphi(t) - \varphi(x)| \leq \delta$, then $|f(t) - f(x)| \leq \omega_\varphi(f, \delta)$. If $|\varphi(t) - \varphi(x)| \geq \delta$, then

$$\begin{aligned} \omega_\varphi(f, |\varphi(t) - \varphi(x)|) &= \omega_\varphi\left(f, \frac{|\varphi(t) - \varphi(x)|}{\delta}\delta\right) \leq \left(1 + \frac{|\varphi(t) - \varphi(x)|}{\delta}\right) \omega_\varphi(f, \delta) \\ &\leq \left(1 + \frac{[\varphi(t) - \varphi(x)]^2}{\delta^2}\right) \omega_\varphi(f, \delta). \quad \square \end{aligned}$$

2. Main part

Theorem 2.1. *Let $A_n : C_\rho(\mathbb{R}_+) \to B_\rho(\mathbb{R}_+)$ be a sequence of positive linear operators such that*

$$\begin{aligned} \left\|A_n\varphi^0 - \varphi^0\right\|_{\rho^0} &= a_n, \\ \left\|A_n\varphi - \varphi\right\|_{\rho^{\frac{1}{2}}} &= b_n, \\ \left\|A_n\varphi^2 - \varphi^2\right\|_{\rho} &= c_n, \end{aligned}$$

where a_n, b_n, c_n tend to zero as n goes to the infinity. Let $\delta_n = \sqrt{a_n + 2b_n + c_n}$ and let η_n be a sequence of real numbers such that

$$\lim_{n\to\infty} \eta_n = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \rho^{\frac{1}{2}}(\eta_n)\delta_n = 0.$$

Then for any function $f \in C_\rho^k(\mathbb{R}_+)$ we have

$$\begin{aligned} (2.1)\ \|A_n f - f\|_\rho \quad &\leq \quad K_f(a_n + c_n) + (\|f\|_\rho + K_f)[\rho^{\frac{1}{2}}(\eta_n)\delta_n\sqrt{1 + a_n} + a_n + \delta_n\sqrt{\delta_n^2 + 4}] \\ &\quad + (2 + a_n)\,\omega_\varphi\left(\frac{f}{\rho}, \rho^{\frac{1}{2}}(\eta_n)\delta_n\right) + 2r_n(3 + 2a_n + 2c_n), \end{aligned}$$

where $K_f = \lim\limits_{x\to\infty} \dfrac{f(x)}{\rho(x)}$ and $r_n = \sup\limits_{x\geq\eta_n} \left|\dfrac{f(x)}{\rho(x)} - K_f\right|$.

Proof Let $f(x)=g(x)+K_f\cdot\rho(x)$. We have $\lim_{x\to\infty} g(x)/\rho(x)=0$ and

$$|A_nf(x)-f(x)|\le|A_ng(x)-g(x)|+K_f\cdot|A_n\rho(x)-\rho(x)|. \tag{2.2}$$

We have, also,

$$r_n=\sup_{x\ge\eta_n}\left|\frac{f(x)}{\rho(x)}-K_f\right|=\sup_{x\ge\eta_n}\frac{|g(x)|}{\rho(x)}.$$

Let $M_n=\max_{x\le\eta_n}|g(x)|$. We consider also the sequences $t_n=\|g\|_\rho\,\rho^{\frac12}(\eta_n)\delta_n(1+a_n)^{-\frac12}>0$ converging to 0 and

$$z_n=\max\Big(\varphi^{-1}(\varphi(\eta_n)+1),\rho^{-1}(\|g\|_\rho\,\rho(\eta_n)/t_n)\Big)>\eta_n.$$

It is easy to see that $\varphi(z_n)-\varphi(\eta_n)\ge1$ and

$$M_n=\max_{x\le\eta_n}|g(x)|\le\max_{x\le\eta_n}\|g\|_\rho\,\rho(x)=\|g\|_\rho\,\rho(\eta_n)\le t_n\rho(z_n).$$

We consider now the functions $h_n\in C_\rho(\mathbb{R}_+)$ defined by

$$h_n(x)=\begin{cases} g(x), & x\le\eta_n,\\ \frac{\varphi(z_n)-\varphi(x)}{\varphi(z_n)-\varphi(\eta_n)}g(\eta_n), & x\in(\eta_n,z_n),\\ 0, & x\ge z_n.\end{cases}$$

Let us prove that

$$|h_n(x)-g(x)|\le2r_n\rho(x),\quad x\ge0. \tag{2.3}$$

Indeed, if $x\le\eta_n$ then the difference $h_n(x)-g(x)$ is 0. If $x\ge\eta_n$, then $|g(x)|\le r_n\rho(x)$. For $x\in[\eta_n,z_n]$ we have

$$|h_n(x)-g(x)|\le|g(x)|+\frac{\varphi(z_n)-\varphi(x)}{\varphi(z_n)-\varphi(\eta_n)}|g(\eta_n)|\le r_n\rho(x)+r_n\rho(\eta_n)\le2r_n\rho(x).$$

For $x\ge z_n$, the difference $|h_n(x)-g(x)|$ is just $|g(x)|$ so less than $2r_n\rho(x)$.
Using relation (2.3) we obtain

$$\begin{aligned}|A_ng(x)-g(x)| &\le A_n(|g(t)-h_n(t)|,x)+|A_nh_n(x)-h_n(x)|+|h_n(x)-g(x)|\\ &\le 2r_n(A_n\rho(x)+\rho(x))+|A_nh_n(x)-h_n(x)|.\end{aligned} \tag{2.4}$$

Next, we give the estimation of the difference $|A_nh_n(x)-h_n(x)|$. For $x\ge z_n$

$$\begin{aligned}|A_nh_n(x)-h_n(x)| &= |A_nh_n(x)|\le A_n|h_n|(x)\le M_nA_n\varphi^0(x)\le t_n\rho(z_n)A_n\varphi^0(x)\\ &\le t_n\rho(x)A_n\varphi^0(x).\end{aligned} \tag{2.5}$$

For $x\le\eta_n$ we get

$$\begin{aligned}|A_nh_n(x)-h_n(x)| &\le |h_n(x)||A_n\varphi^0(x)-\varphi^0(x)|+A_n(|h_n(t)-h_n(x)|,x)\\ &\le |g(x)||A_n\varphi^0(x)-\varphi^0(x)|+A_n(|h_n(t)-g(t)|,x)+A_n(|g(t)-g(x)|,x)\\ &\le \|g\|_\rho\,\rho(x)|A_n\varphi^0(x)-\varphi^0(x)|+2r_nA_n\rho(x)+A_n(|g(t)-g(x)|,x)\end{aligned} \tag{2.6}$$

Using the relation

$$\begin{aligned}|g(t)-g(x)| &\le \frac{|g(t)|}{\rho(t)}|\rho(t)-\rho(x)|+\rho(x)\left|\frac{g(t)}{\rho(t)}-\frac{g(x)}{\rho(x)}\right|\\ &\le \|g\|_\rho\,|\rho(t)-\rho(x)|+\rho(x)\left|\frac{g}{\rho}(t)-\frac{g}{\rho}(x)\right|\\ &\le \|g\|_\rho\,|\rho(t)-\rho(x)|+\rho(x)\left[1+\frac{(\varphi(t)-\varphi(x))^2}{h^2}\right]\omega_\varphi\left(\frac{g}{\rho},h\right)\end{aligned} \tag{2.7}$$

we obtain

$$
\begin{aligned}
(2.8) \qquad A_n(|g(t)-g(x)|,x) &\leq \|g\|_\rho A_n(|\rho(t)-\rho(x)|,x)+\rho(x)A_n\varphi^0(x)\,\omega_\varphi\left(\frac{g}{\rho},h\right)\\
&\quad +\rho(\eta_n)\frac{A_n([\varphi(t)-\varphi(x)]^2,x)}{h^2}\omega_\varphi\left(\frac{g}{\rho},h\right).
\end{aligned}
$$

In the case $x\in[\eta_n,z_n]$ we have

$$
\begin{aligned}
|A_nh_n(x)-h_n(x)| &\leq |h_n(x)||A_n\varphi^0(x)-\varphi^0(x)|+A_n(|h_n(t)-h_n(x)|,x)\\
(2.9) \qquad &\leq |g(\eta_n)||A_n\varphi^0(x)-\varphi^0(x)|+A_n(|h_n(t)-h_n(x)|,x)\\
&\leq \|g\|_\rho\,\rho(x)|A_n\varphi^0(x)-\varphi^0(x)|+A_n(|h_n(t)-h_n(x)|,x).
\end{aligned}
$$

We estimate now the difference $|h_n(t)-h_n(x)|$. For $t\geq z_n$ we have

$$
|h_n(t)-h_n(x)|=|h_n(x)|=\frac{\varphi(z_n)-\varphi(x)}{\varphi(z_n)-\varphi(\eta_n)}|g(\eta_n)|\leq\|g\|_\rho\,\rho^{\frac12}(x)\rho^{\frac12}(\eta_n)|\varphi(t)-\varphi(x)|.
$$

For $t\in[\eta_n,z_n]$ we have

$$
|h_n(t)-h_n(x)|=\left|\frac{\varphi(z_n)-\varphi(t)}{\varphi(z_n)-\varphi(\eta_n)}g(\eta_n)-\frac{\varphi(z_n)-\varphi(x)}{\varphi(z_n)-\varphi(\eta_n)}g(\eta_n)\right|\leq\|g\|_\rho\,\rho^{\frac12}(x)\rho^{\frac12}(\eta_n)|\varphi(t)-\varphi(x)|.
$$

For $t\leq\eta_n$, we use the relation (2.7) and we obtain

$$
\begin{aligned}
|h_n(t)-h_n(x)| &= \left|g(t)-\frac{\varphi(z_n)-\varphi(x)}{\varphi(z_n)-\varphi(\eta_n)}g(\eta_n)\right|\leq|g(t)-g(\eta_n)|+|g(\eta_n)||\varphi(x)-\varphi(\eta_n)|\\
&\leq \|g\|_\rho\,|\rho(t)-\rho(\eta_n)|+\rho(\eta_n)\left[1+\frac{[\varphi(t)-\varphi(\eta_n)]^2}{h^2}\right]\omega_\varphi\left(\frac{g}{\rho},h\right)\\
&\quad +\|g\|_\rho\,\rho^{\frac12}(x)\rho^{\frac12}(\eta_n)|\varphi(t)-\varphi(x)|\\
&\leq \|g\|_\rho\,|\rho(t)-\rho(x)|+\left[\rho(x)+\rho(\eta_n)\frac{[\varphi(t)-\varphi(x)]^2}{h^2}\right]\omega_\varphi\left(\frac{g}{\rho},h\right)\\
&\quad +\|g\|_\rho\,\rho^{\frac12}(x)\rho^{\frac12}(\eta_n)|\varphi(t)-\varphi(x)|.
\end{aligned}
$$

We can deduce

$$
\begin{aligned}
A_n(|h_n(t)-h_n(x)|,x) &\leq \|g\|_\rho A_n(|\rho(t)-\rho(x)|,x)+\rho^{\frac12}(x)\rho^{\frac12}(\eta_n)\,\|g\|_\rho A_n(|\varphi(t)-\varphi(x)|,x)\\
(2.10) \qquad &\quad +\left[\rho(x)A_n\varphi^0(x)+\rho(\eta_n)\frac{A_n([\varphi(t)-\varphi(x)]^2,x)}{h^2}\right]\omega_\varphi\left(\frac{g}{\rho},h\right).
\end{aligned}
$$

To obtain the final result we use the relations

$$\begin{aligned}
|A_n\rho(x)-\rho(x)| &\le |A_n\varphi^0(x)-\varphi^0(x)|+|A_n\varphi^2(x)-\varphi^2(x)|\\
&\le a_n+\rho(x)c_n,\\
A_n\rho(x)+\rho(x) &\le |A_n\rho(x)-\rho(x)|+2\rho(x)\\
&\le \rho(x)(a_n+c_n+2),\\
A_n\left([\varphi(t)-\varphi(x)]^2,x\right) &= A_n\varphi^2(x)-2\varphi(x)A_n\varphi(x)+\varphi^2(x)A_n\varphi^0(x)\\
&= A_n\varphi^2(x)-\varphi^2(x)-2\varphi(x)[A_n\varphi(x)-\varphi(x)]+\varphi^2(x)[A_n\varphi^0(x)-1]\\
&\le \rho(x)c_n+2\varphi(x)\rho^{\frac{1}{2}}(x)b_n+\varphi^2(x)a_n\\
&\le \rho(x)\delta_n^2,\\
A_n(|\varphi(t)-\varphi(x)|,x) &\le \sqrt{A_n\left([\varphi(t)-\varphi(x)]^2,x\right)}\sqrt{A_n\varphi^0(x)}\\
&\le \rho^{\frac{1}{2}}(x)\delta_n\sqrt{1+a_n},\\
A_n(|\rho(t)-\rho(x)|,x) &= A_n(|\varphi(t)-\varphi(x)|\cdot|\varphi(t)+\varphi(x)|,x)\\
&\le \sqrt{A_n\left([\varphi(t)-\varphi(x)]^2,x\right)}\sqrt{A_n\left([\varphi(t)+\varphi(x)]^2,x\right)}\\
&\le \rho(x)\delta_n\sqrt{\delta_n^2+4}.
\end{aligned}$$

From the estimations (2.5), (2.6) and (2.9) we deduce

$$\begin{aligned}
\frac{A_n(|h_n(t)-h_n(x)|,x)}{\rho(x)} &\le \|g\|_\rho\,\rho^{\frac{1}{2}}(\eta_n)\delta_n\sqrt{1+a_n}+\|g\|_\rho\,a_n+\|g\|_\rho\,\delta_n\sqrt{\delta_n^2+4}\\
&\quad+\left(1+a_n+\frac{\rho(\eta_n)\delta_n^2}{h^2}\right)\omega_\varphi\left(\frac{g}{\rho},h\right)+2r_n(1+a_n+c_n).
\end{aligned}$$

Considering $h=\rho^{\frac{1}{2}}(\eta_n)\delta_n$ in the above inequality, the relations (2.2) and (2.4) give us

$$\begin{aligned}
\frac{|A_nf(x)-f(x)|}{\rho(x)} &\le K_f(a_n+c_n)+\|g\|_\rho\,[\rho^{\frac{1}{2}}(\eta_n)\delta_n\sqrt{1+a_n}+a_n+\delta_n\sqrt{\delta_n^2+4}]\\
&\quad+(2+a_n)\,\omega_\varphi\left(\frac{g}{\rho},\rho^{\frac{1}{2}}(\eta_n)\delta_n\right)+2r_n(3+2a_n+2c_n).
\end{aligned}$$

Because $\omega_\varphi\left(\frac{f}{\rho},h\right)=\omega_\varphi\left(\frac{g}{\rho},h\right)$ and $\|g\|_\rho\le\|f\|_\rho+K_f$ we obtain the estimation

$$\begin{aligned}
\|A_nf-f\|_\rho &\le K_f(a_n+c_n)+(\|f\|_\rho+K_f)[\rho^{\frac{1}{2}}(\eta_n)\delta_n\sqrt{1+a_n}+a_n+\delta_n\sqrt{\delta_n^2+4}]\\
&\quad+(2+a_n)\,\omega_\varphi\left(\frac{f}{\rho},\rho^{\frac{1}{2}}(\eta_n)\delta_n\right)+2r_n(3+2a_n+2c_n).
\end{aligned}$$

□

Remark 2.1. *The estimation (2.1) of A_nf-f in ρ-norm is in terms of the sequences converging to 0: $a_n,b_n,c_n,\rho^{\frac{1}{2}}(\eta_n)\delta_n$, $\omega_\varphi\left(\frac{f}{\rho},\rho^{\frac{1}{2}}(\eta_n)\delta_n\right)$ and r_n. Indeed, $\omega_\varphi\left(\frac{f}{\rho},\rho^{\frac{1}{2}}(\eta_n)\delta_n\right)$ tends to 0, because f/ρ is uniformly continuous. Because $\eta_n\to\infty$ and because $\lim_{x\to\infty}f(x)/\rho(x)=K_f$, we obtain $r_n=\sup_{x\ge\eta_n}|f(x)/\rho(x)-K_f|\to 0$.*

Example 2.1. *For $\varphi(x)=x$ and the Szasz-Mirakjan operators $M_n:C_\rho(\mathbb{R}_+)\to B_\rho(\mathbb{R}_+)$ defined by*

$$M_nf(x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right),$$

which have the properties $M_n 1(x) = 1$, $M_n e_1(x) = x$ *and* $M_n e_2(x) = x^2 + x/n$ *(so,* $a_n = b_n = 0$ *and* $c_n = 1/(2n)$*), we obtain for every function* $f : \mathbb{R}_+ \to \mathbb{R}$ *with* $\lim_{x\to\infty} \frac{f(x)}{1+x^2} = K_f < \infty$ *the estimation*

$$\sup_{x\geq 0} \frac{|M_n f(x) - f(x)|}{1+x^2} \leq \frac{K_f}{2n} + (\|f\|_\rho + K_f)\left(\frac{3}{2\sqrt{n}} + \sqrt{\frac{1+\eta_n^2}{2n}}\right) + 2\omega\left(\frac{f}{\rho}, \sqrt{\frac{1+\eta_n^2}{2n}}\right) + 8\, r_n,$$

where $\eta_n \to \infty$ *and* $r_n = \sup_{x\geq\eta_n} \left|\frac{f(x)}{1+x^2} - K_f\right| \to 0$.

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