THE RATE OF CONVERGENCE OF POSITIVE LINEAR OPERATORS IN WEIGHTED SPACES

Abstract. We estimate the rate of approximation of positive linear operators for unbounded functions defined on the positive semi-axis, in terms of the modulus of continuity of the first order and the rate of convergence of the function toward infinity.

1. Introduction

Let \( R^+ = [0, \infty) \) and let \( \varphi : R^+ \to R^+ \) be an unbounded strictly increasing continuous function with \( \varphi(0) = 0 \) and with the property that \( \varphi^{-1} \) is uniformly continuous. Let \( \rho(x) = 1 + \varphi^2(x) \) be a weight function and let \( B_\rho(R^+) \) be the Banach space defined by

\[
B_\rho(R^+) = \{ f : R^+ \to R \mid \text{there exists } M > 0 \text{ such that } |f(x)| \leq M \cdot \rho(x), \text{ for all } x \geq 0 \}.
\]

This weighted space can be endowed with the \( \rho \)-norm

\[
\|f\|_\rho = \sup_{x \geq 0} \frac{|f(x)|}{\rho(x)}.
\]

We define also the subspaces

\[
C_\rho(R^+) = \{ f \in B_\rho(R^+), f \text{ is continuous} \},
\]

\[
C^k_\rho(R^+) = \left\{ f \in C_\rho(R^+), \lim_{x \to +\infty} \frac{f(x)}{\rho(x)} = K_f < +\infty \right\}.
\]

In [1] is given the following Korovkin-type theorem

**Theorem 1.1.** If \( A_n : C_\rho(R) \to B_\rho(R) \) is a sequence of positive linear operators such that

\[
\lim_{n \to \infty} \|A_n \varphi^i - \varphi^i\|_\rho = 0, \quad i = 0, 1, 2,
\]

then for any function \( f \in C^k_\rho(R) \) we have

\[
\lim_{n \to \infty} \|A_n f - f\|_\rho = 0.
\]

In [2] it is given an estimation of the rate of convergence for positive linear operators of the following type

\[
B_n f(x) = \left\{ \begin{array}{ll} A_n f(x), & x \leq \eta_n, \\ f(x), & x > \eta_n, \end{array} \right.
\]

where \( (A_n)_{n \in \mathbb{N}} \) is a sequence of positive linear operators acting from \( C_\rho(R^+) \) to \( B_\rho(R^+) \) and \( (\eta_n)_{n \in \mathbb{N}} \) is a sequence converging to infinity when \( n \) approaches infinity. The result is the following

Date: November 6, 2008.

1991 Mathematics Subject Classification. 41A25, 41A36.

Key words and phrases. weighted spaces, rate of approximation, positive linear operators.
Theorem 1.2. Let $A_n : C_p(\mathbb{R}_+) \to B_p(\mathbb{R}_+)$ be a sequence of positive linear operators with
\[
\|A_n \varphi^0 - \varphi^0\|_\rho = a_n, \\
\|A_n \varphi - \varphi\|_{\rho^{\frac{1}{2}}} = b_n, \\
\|A_n \varphi^2 - \varphi^2\|_\rho = c_n, \\
\|A_n \varphi^3 - \varphi^3\|_{\rho^{\frac{1}{2}}} = d_n,
\]
where $a_n, b_n, c_n$ and $d_n$ tend to zero as $n$ goes to the infinity. Let $\eta_n$ be a sequence of real numbers such that
\[
\lim_{n \to \infty} \eta_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \rho^{\frac{1}{2}}(\eta_n)\delta_n = 0,
\]
where $\delta_n = \sqrt{(a_n + 2b_n + c_n)(1 + a_n) + a_n + 3b_n + 3c_n + d_n}$. Then for every $f \in C_p(\mathbb{R}_+)$
\[
\sup_{0 \leq x \leq \eta_n} \frac{|A_n f(x) - f(x)|}{\rho(x)} \leq (7 + 4a_n + 2c_n) \cdot \omega_\varphi \left( f, \rho^{\frac{1}{2}}(\eta_n)\delta_n \right) + \|f\|_{\rho} a_n,
\]
where $\omega_\varphi$ is defined for $f \in C_p(\mathbb{R}_+)$ and $\delta \geq 0$ by
\[
\omega_\varphi (f, \delta) = \sup_{x,y \geq 0} \frac{|f(x) - f(y)|}{\rho(x) + \rho(y)} \quad \text{for} \quad |\varphi(x) - \varphi(y)| \leq \delta.
\]

In the same paper it is proved that if $\varphi$ satisfies also the condition
\[
|x - y| \leq M |\varphi(x) - \varphi(y)|^\alpha,
\]
for every $x, y \geq 0$, where $M > 0$ and $\alpha \in (0, 1]$, then the right-hand side of the estimation from the above theorem tends to 0.

In the present paper, we want to give an estimation of the rate of convergence of $A_n f$ toward $f$ in the general case of any sequence of positive linear operators $A_n : C_p(\mathbb{R}_+) \to B_p(\mathbb{R}_+)$. For this, we need the following modulus of continuity
\[
\omega_\varphi (f, \delta) = \sup_{x,t \geq 0} \frac{|f(x) - f(t)|}{\rho(x) - \rho(t)} \quad \text{for} \quad |\varphi(x) - \varphi(t)| \leq \delta
\]
defined for every bounded function $f : \mathbb{R}_+ \to \mathbb{R}$ and every $\delta \geq 0$. This modulus is an nonnegative, increasing, bounded function in $\delta$ and has the following properties

**Proposition 1.1.** For every bounded and uniformly continuous function $f : \mathbb{R}_+ \to \mathbb{R}$ we have
\[
\lim_{n \to \infty} \omega_\varphi (f, \delta_n) = 0 \quad \text{whenever} \quad \delta_n \to 0.
\]

**Proof** This property is true due to the property of $\varphi^{-1}$ to be uniformly continuous and due to the following representation
\[
\omega_\varphi (f, \delta) = \omega (f \circ \varphi^{-1}, \delta).
\]
Indeed, $f \circ \varphi^{-1}$ is a bounded uniformly continuous function and the usual modulus of continuity $\omega$ is continuous in 0 for such a function. □

**Remark 1.1.** If $\varphi(x) = x$, then $\omega_\varphi$ reduces to the usual modulus of continuity. If $\varphi^{-1}$ is a Holder function, i.e. there exist $M > 0$ and $\alpha \in (0, 1]$ such that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ has the property
\[
|x - y| \leq M |\varphi(x) - \varphi(y)|^\alpha,
\]
for every $x, y \geq 0$,

then we have the relation
\[
\omega_\varphi (f, \delta) \leq \omega (f, M\delta^\alpha).
\]
Proposition 1.2. For every bounded function \( f : \mathbb{R}_+ \to \mathbb{R} \) and for every \( t, x, \delta \geq 0 \) we have
\[
|f(t) - f(x)| \leq \left( 1 + \frac{[\varphi(t) - \varphi(x)]^2}{\delta^2} \right) \omega_{\varphi}(f, \delta).
\]

Proof We prove first, that for every \( m \in \mathbb{N} \)
\[
\omega_{\varphi}(f, m\delta) \leq m \cdot \omega_{\varphi}(f, \delta).
\]
For \( m = 0 \) and \( m = 1 \) it holds with equality. For \( m \geq 2 \), let \( t > x \geq 0 \) such that we have \( \varphi(t) - \varphi(x) \leq m\delta \). We construct the points \( x_0 = x < x_1 < \cdots < x_m = t \) with the property
\[
\varphi(x_k) - \varphi(x_{k-1}) = \frac{\varphi(t) - \varphi(x)}{m} \leq \delta.
\]
We obtain
\[
|f(t) - f(x)| \leq \sum_{k=1}^{m} |f(x_k) - f(x_{k-1})| \leq \sum_{k=1}^{m} \omega_{\varphi}(f, \delta) \leq m \cdot \omega_{\varphi}(f, \delta).
\]
This proves the relation \( \omega_{\varphi}(f, m\delta) \leq m \cdot \omega_{\varphi}(f, \delta) \). Considering \( \lambda > 0 \) we obtain
\[
\omega_{\varphi}(f, \lambda\delta) \leq \omega_{\varphi}(f, (\lfloor \lambda \rfloor + 1)\delta) \leq (\lfloor \lambda \rfloor + 1)\omega_{\varphi}(f, \delta) \leq (\lambda + 1)\omega_{\varphi}(f, \delta).
\]
Now, we can prove the relation from the proposition. Let \( t, x \geq 0 \). We have
\[
|f(t) - f(x)| \leq \omega_{\varphi}(f, |\varphi(t) - \varphi(x)|).
\]
If \( |\varphi(t) - \varphi(x)| \leq \delta \), then \( |f(t) - f(x)| \leq \omega_{\varphi}(f, \delta) \). If \( |\varphi(t) - \varphi(x)| \geq \delta \), then
\[
\omega_{\varphi}(f, |\varphi(t) - \varphi(x)|) = \omega_{\varphi}\left(f, \frac{|\varphi(t) - \varphi(x)|}{\delta}\right) \leq \left( 1 + \frac{|\varphi(t) - \varphi(x)|}{\delta} \right) \omega_{\varphi}(f, \delta)
\]
\[
\leq \left( 1 + \frac{|\varphi(t) - \varphi(x)|^2}{\delta^2} \right) \omega_{\varphi}(f, \delta).
\]
\[\square\]

2. Main part

Theorem 2.1. Let \( A_n : C_{\varphi}(\mathbb{R}_+) \to B_{\rho}(\mathbb{R}_+) \) be a sequence of positive linear operators such that
\[
\|A_n \varphi^0 - \varphi^0\|_{\rho} = a_n,
\]
\[
\|A_n \varphi - \varphi\|_{\rho^2} = b_n,
\]
\[
\|A_n \varphi^2 - \varphi^2\|_{\rho} = c_n,
\]
where \( a_n, b_n, c_n \) tend to zero as \( n \) goes to the infinity. Let \( \delta_n = \sqrt{a_n + 2b_n + c_n} \) and let \( \eta_n \) be a sequence of real numbers such that
\[
\lim_{n \to \infty} \eta_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \rho^{\frac{1}{2}}(\eta_n)\delta_n = 0.
\]
Then for any function \( f \in C_{\varphi}^{k}(\mathbb{R}_+) \) we have
\[
(2.1) \|A_n f - f\|_{\rho} \leq K_f(a_n + c_n) + (\|f\|_{\rho} + K_f)[\rho^{\frac{1}{2}}(\eta_n)\delta_n \sqrt{1 + a_n + a_n + \delta_n \sqrt{b_n^2 + 4}}]
\]
\[
+ (2 + a_n + \delta_n) \omega_{\varphi}\left( \frac{f}{\rho}, \rho^{\frac{1}{2}}(\eta_n)\delta_n \right) + 2r_n(3 + 2a_n + 2c_n),
\]
where \( K_f = \lim_{x \to \infty} \frac{f(x)}{\rho(x)} \) and \( r_n = \sup_{x \geq \eta_n} \left| \frac{f(x)}{\rho(x)} - K_f \right| \).
Proof Let \( f(x) = g(x) + K_f \cdot \rho(x) \). We have \( \lim_{x \to \infty} g(x) / \rho(x) = 0 \) and
\[
|A_n f(x) - f(x)| \leq |A_n g(x) - g(x)| + K_f \cdot |A_n \rho(x) - \rho(x)|.
\]

We have, also,
\[
r_n = \sup_{x \geq \eta_n} \left| \frac{f(x)}{\rho(x)} - K_f \right| = \sup_{x \geq \eta_n} \frac{|g(x)|}{\rho(x)}.
\]

Let \( M_n = \max_{x \leq \eta_n} |g(x)| \). We consider also the sequences \( t_n = \|g\|_\rho \cdot \rho^{\frac{1}{2}}(\eta_n)\delta_n (1 + a_n)^{-\frac{1}{2}} > 0 \)
converging to 0 and
\[
z_n = \max \left( \varphi^{-1}(\varphi(\eta_n) + 1), \rho^{-1}(\|g\|_\rho(\eta_n)/t_n) \right) > \eta_n.
\]

It is easy to see that \( \varphi(z_n) - \varphi(\eta_n) \geq 1 \) and
\[
M_n = \max_{x \leq \eta_n} |g(x)| \leq \max_{x \leq \eta_n} \|g\|_\rho \rho(x) = \|g\|_\rho \rho(\eta_n) \leq t_n \rho(z_n).
\]

We consider now the functions \( h_n \in C_\rho(\mathbb{R}_+) \) defined by
\[
h_n(x) = \begin{cases} 
g(x), & x \leq \eta_n, \\ \frac{\varphi(z_n) - \varphi(x)}{\varphi(z_n) - \varphi(\eta_n)} g(\eta_n), & x \in (\eta_n, z_n), \\ 0, & x \geq z_n.
\end{cases}
\]

Let us prove that
\[
|h_n(x) - g(x)| \leq 2r_n \rho(x), \quad x \geq 0.
\]
Indeed, if \( x \leq \eta_n \) then the difference \( h_n(x) - g(x) \) is 0. If \( x \geq \eta_n \), then \( |g(x)| \leq r_n \rho(x) \). For \( x \in [\eta_n, z_n] \) we have
\[
|h_n(x) - g(x)| \leq |g(x)| + \frac{\varphi(z_n) - \varphi(x)}{\varphi(z_n) - \varphi(\eta_n)} |g(\eta_n)| \leq r_n \rho(x) + r_n \rho(\eta_n) \leq 2r_n \rho(x).
\]

For \( x \geq z_n \), the difference \( |h_n(x) - g(x)| \) is just \( |g(x)| \) so less than \( 2r_n \rho(x) \).

Using relation (2.3) we obtain
\[
|A_n g(x) - g(x)| \leq A_n |g(t) - h_n(t)|, x + |A_n h_n(x) - h_n(x)| + |h_n(x) - g(x)|
\leq 2r_n (A_n \rho(x) + \rho(x)) + |A_n h_n(x) - h_n(x)|.
\]

Next, we give the estimation of the difference \( |A_n h_n(x) - h_n(x)| \). For \( x \geq z_n \)
\[
|A_n h_n(x) - h_n(x)| = |A_n h_n(x) - h_n(x)| \leq A_n |h_n(x)| \leq M_n A_n \varphi^0(x) \leq t_n \rho(z_n) A_n \varphi^0(x)
\leq t_n \rho(x) A_n \varphi^0(x).
\]

For \( x \leq \eta_n \) we get
\[
|A_n h_n(x) - h_n(x)| \leq |h_n(x)| |A_n \varphi^0(x) - \varphi^0(x)| + A_n (|h_n(t) - h_n(x)|, x)
\leq |g(x)| |A_n \varphi^0(x) - \varphi^0(x)| + A_n (|h_n(t) - g(t)|, x) + A_n (|g(t) - g(x)|, x)
\leq \|g\|_\rho \rho(x) A_n \varphi^0(x) - \varphi^0(x)| + 2r_n A_n \rho(x) + A_n (|g(t) - g(x)|, x)
\]

Using the relation
\[
|g(t) - g(x)| \leq \left| \frac{g(t)}{\rho(t)} |\rho(t) - \rho(x)| \right| + \rho(x) \left| \frac{g(t)}{\rho(t)} - \frac{g(x)}{\rho(x)} \right|
\leq \|g\|_\rho |\rho(t) - \rho(x)| + \rho(x) \left| \frac{g(t)}{\rho(t)} - \frac{g(x)}{\rho(x)} \right|
\leq \|g\|_\rho |\rho(t) - \rho(x)| + \rho(x) \left[ 1 + \frac{(\varphi(t) - \varphi(x))^2}{h^2} \right] \omega_{\varphi}(\frac{g}{\rho}, h)
\]

(2.7)
we obtain

\begin{equation}
A_n(|g(t) - g(x)|, x) \leq \|g\|_\rho A_n(|\rho(t) - \rho(x)|, x) + \rho(x)A_n\varphi^0(x) \omega_\varphi \left( \frac{g}{\rho}, h \right) + \rho(\eta) \frac{A_n(|\varphi(t) - \varphi(x)|^2, x)}{h^2} \omega_\varphi \left( \frac{g}{\rho}, h \right).
\end{equation}

In the case \( x \in [\eta_n, z_n] \) we have

\begin{equation}
|A_n h_n(x) - h_n(x)| \leq |h_n(x)||A_n\varphi^0(x) - \varphi^0(x)| + A_n(|h_n(t) - h_n(x)|, x)
\end{equation}

\begin{equation}
\leq |g(\eta_n)||A_n\varphi^0(x) - \varphi^0(x)| + A_n(|h_n(t) - h_n(x)|, x)
\end{equation}

\begin{equation}
\leq \|g\|_\rho \|\rho(x)A_n\varphi^0(x) - \varphi^0(x)| + A_n(|h_n(t) - h_n(x)|, x).
\end{equation}

We estimate now the difference \( |h_n(t) - h_n(x)| \). For \( t \geq z_n \) we have

\begin{equation}
|h_n(t) - h_n(x)| = |h_n(x)| = \frac{\varphi(z_n) - \varphi(x)}{\varphi(z_n) - \varphi(\eta_n)} g(\eta_n) \leq \|g\|_\rho \frac{1}{\rho} \frac{1}{\rho}(\eta_n)|\varphi(t) - \varphi(x)|.
\end{equation}

For \( t \in [\eta_n, z_n] \) we have

\begin{equation}
|h_n(t) - h_n(x)| = \left| \frac{\varphi(z_n) - \varphi(t)}{\varphi(z_n) - \varphi(\eta_n)} g(\eta_n) - \frac{\varphi(z_n) - \varphi(x)}{\varphi(z_n) - \varphi(\eta_n)} g(\eta_n) \right| \leq \|g\|_\rho \frac{1}{\rho} \frac{1}{\rho}(\eta_n)|\varphi(t) - \varphi(x)|.
\end{equation}

For \( t \leq \eta_n \), we use the relation (2.7) and we obtain

\begin{equation}
|h_n(t) - h_n(x)| = \left| g(t) - \frac{\varphi(z_n) - \varphi(x)}{\varphi(z_n) - \varphi(\eta_n)} g(\eta_n) \right| \leq |g(t) - g(\eta_n)| + |g(\eta_n)||\varphi(x) - \varphi(\eta_n)|
\end{equation}

\begin{equation}
\leq \|g\|_\rho |\rho(t) - \rho(\eta_n)| + \rho(\eta_n) \left[ 1 + \frac{|\varphi(t) - \varphi(\eta_n)|^2}{h^2} \right] \omega_\varphi \left( \frac{g}{\rho}, h \right)
\end{equation}

\begin{equation}
+ \|g\|_\rho \frac{1}{\rho}(\eta_n)|\varphi(t) - \varphi(x)|
\end{equation}

\begin{equation}
\leq \|g\|_\rho |\rho(t) - \rho(\eta_n)| + \rho(\eta_n) \left[ \frac{|\varphi(t) - \varphi(x)|^2}{h^2} \right] \omega_\varphi \left( \frac{g}{\rho}, h \right)
\end{equation}

\begin{equation}
+ \|g\|_\rho \frac{1}{\rho}(\eta_n)|\varphi(t) - \varphi(x)|.
\end{equation}

We can deduce

\begin{equation}
A_n(|h_n(t) - h_n(x)|, x) \leq \|g\|_\rho A_n(|\rho(t) - \rho(x)|, x) + \rho(\eta) \frac{1}{\rho}(\eta_n) \|g\|_\rho A_n(|\varphi(t) - \varphi(x)|, x)
\end{equation}

\begin{equation}
+ \left[ \rho(x)A_n\varphi^0(x) + \rho(\eta)A_n\left[ \frac{|\varphi(t) - \varphi(x)|^2}{h^2} \right] \right] \omega_\varphi \left( \frac{g}{\rho}, h \right).
\end{equation}
To obtain the final result we use the relations
\[
|A_n\rho(x) - \rho(x)| \leq |A_n\varphi^0(x) - \varphi^0(x)| + |A_n\varphi^2(x) - \varphi^2(x)|
\]
\[
\leq a_n + \rho(x)c_n,
\]
\[
A_n\rho(x) + \rho(x) \leq |A_n\rho(x) - \rho(x)| + 2\rho(x)
\]
\[
\leq \rho(x)(a_n + c_n + 2),
\]
\[
A_n\left(\|\varphi(t) - \varphi(x)\|^2, x\right) = A_n\varphi^2(x) - 2\varphi(x)A_n\varphi(x) + \varphi^2(x)A_n\varphi^0(x)
\]
\[
= A_n\varphi^2(x) - \varphi^2(x) - 2\varphi(x)[A_n\varphi(x) - \varphi(x)] + \varphi^2(x)[A_n\varphi^0(x) - 1]
\]
\[
\leq \rho(x)c_n + 2\varphi(x)\rho^\frac{1}{2}(x)b_n + \varphi^2(x)a_n
\]
\[
\leq \rho(x)\delta_n^2,
\]
\[
A_n(\|\varphi(t) - \varphi(x)\|, x) \leq \sqrt{A_n(\|\varphi(t) - \varphi(x)\|^2, x)}\sqrt{A_n\varphi^0(x)}
\]
\[
\leq \rho^\frac{1}{2}(x)\delta_n\sqrt{1 + a_n},
\]
\[
A_n(\|\rho(t) - \rho(x)\|, x) = A_n(\|\varphi(t) - \varphi(x)\| \cdot |\varphi(t) + \varphi(x)|, x)
\]
\[
\leq \sqrt{A_n(\|\varphi(t) - \varphi(x)\|^2, x)}\sqrt{A_n(\|\varphi(t) + \varphi(x)\|^2, x)}
\]
\[
\leq \rho(x)\delta_n\sqrt{\delta_n^2 + 4}.
\]

From the estimations (2.5), (2.6) and (2.9) we deduce
\[
\frac{A_n(h_n(t) - h_n(x), x)}{\rho(x)} \leq \|g\|_{\rho} \rho^\frac{1}{2}(\eta_n)\delta_n\sqrt{1 + a_n} + \|g\|_{\rho} a_n + \|g\|_{\rho} \delta_n \sqrt{\delta_n^2 + 4}
\]
\[
+ \left(1 + a_n + \frac{\rho(\eta_n)\delta_n^2}{h^2}\right)\omega_{\frac{g}{\rho}}\left(\frac{g}{\rho}, h\right) + 2r_n(1 + a_n + c_n).
\]

Considering \(h = \rho^\frac{1}{2}(\eta_n)\delta_n\) in the above inequality, the relations (2.2) and (2.4) give us
\[
\frac{|A_n f(x) - f(x)|}{\rho(x)} \leq K_f(a_n + c_n) + \|g\|_{\rho} \left[\rho^\frac{1}{2}(\eta_n)\delta_n\sqrt{1 + a_n} + a_n + \delta_n \sqrt{\delta_n^2 + 4}\right]
\]
\[
+ (2 + a_n)\omega_{\frac{g}{\rho}}\left(\frac{g}{\rho}, \rho^\frac{1}{2}(\eta_n)\delta_n\right) + 2r_n(3 + 2a_n + 2c_n).
\]

Because \(\omega_{\frac{f}{\rho}}\left(\frac{f}{\rho}, h\right) = \omega_{\frac{g}{\rho}}\left(\frac{g}{\rho}, h\right)\) and \(\|g\|_{\rho} \leq \|f\|_{\rho} + K_f\) we obtain the estimation
\[
\|A_n f - f\|_{\rho} \leq K_f(a_n + c_n) + (\|f\|_{\rho} + K_f)\rho^\frac{1}{2}(\eta_n)\delta_n\sqrt{1 + a_n} + a_n + \delta_n \sqrt{\delta_n^2 + 4}
\]
\[
+ (2 + a_n)\omega_{\frac{f}{\rho}}\left(\frac{f}{\rho}, \rho^\frac{1}{2}(\eta_n)\delta_n\right) + 2r_n(3 + 2a_n + 2c_n).
\]

\[\square\]

**Remark 2.1.** The estimation (2.4) of \(A_n f - f\) in \(\rho\)-norm is in terms of the sequences converging to 0: \(a_n, b_n, c_n, \rho^\frac{1}{2}(\eta_n)\delta_n, \omega_{\frac{f}{\rho}}\left(\frac{f}{\rho}, \rho^\frac{1}{2}(\eta_n)\delta_n\right)\) and \(r_n\). Indeed, \(\omega_{\frac{f}{\rho}}\left(\frac{f}{\rho}, \rho^\frac{1}{2}(\eta_n)\delta_n\right)\) tends to 0, because \(f/\rho\) is uniformly continuous. Because \(\eta_n \to \infty\) and because \(\lim_{x \to \infty} f(x)/\rho(x) = K_f\), we obtain \(r_n = \sup_{x \geq \eta_n} |f(x)/\rho(x) - K_f| \to 0\).

**Example 2.1.** For \(\varphi(x) = x\) and the Szasz-Mirakjan operators \(M_n : C_\rho(\mathbb{R}_+) \to B_\rho(\mathbb{R}_+)\) defined by
\[
M_n f(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),
\]
which have the properties $M_n 1(x) = 1$, $M_n e_1(x) = x$ and $M_n e_2(x) = x^2 + x/n$ (so, $a_n = b_n = 0$ and $c_n = 1/(2n)$), we obtain for every function $f : \mathbb{R}_+ \to \mathbb{R}$ with $\lim_{x \to \infty} \frac{f(x)}{1 + x^2} = K_f < \infty$ the estimation
\[
\sup_{x \geq 0} \frac{|M_n f(x) - f(x)|}{1 + x^2} \leq \frac{K_f}{2n} + (\|f\|_\rho + K_f) \left( \frac{3}{2\sqrt{n}} + \sqrt{\frac{1 + \eta_n^2}{2n}} \right) + 2\omega \left( \frac{f}{\rho}, \sqrt{\frac{1 + \eta_n^2}{2n}} \right) + 8 r_n,
\]
where $\eta_n \to \infty$ and $r_n = \sup_{x \geq \eta_n} \left| \frac{f(x)}{1 + x^2} - K_f \right| \to 0$.

References


Technical University of Cluj-Napoca, C. Daicoviciu, nr. 15, Cluj-Napoca, Romania
E-mail address: adrian.holhos@math.utcluj.ro