Uniform approximation by positive linear operators on noncompact intervals

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Abstract

We characterize the functions defined on a noncompact interval, which are uniformly approximated by a sequence of positive linear operators. The particular cases of the Szász-Mirakjan operators, the Baskakov operators, the Meier-König and Zeller operators, the Gauss-Weierstrass operators and the Bleimann-Butzer-Hahn operators are included.

1 Introduction

Let $C(I)$ be the space of continuous functions defined on a noncompact interval $I \subseteq \mathbb{R}$. For $n \in \mathbb{N}$, let $A_n \colon C(I) \to C(I)$ be a sequence of positive linear operators. We want to characterize those functions $f \in C(I)$ which can be uniformly approximated by these operators, i.e.

$$\sup_{x \in I} |A_n(f,x) - f(x)| = \|A_n f - f\|_\infty \to 0.$$

An important result in this direction is Theorem 2 from [7], which gives the necessary and sufficient conditions for uniform convergence, when the functions are continuous and bounded on I. Using the Ditzian-Totik modulus of smoothness, the result is proved for a large class of positive linear operators preserving constant functions and satisfying certain assumptions. In [5], the authors study what happens if the boundedness assumption on the functions is dropped, and give some results for Bernstein-type operators, using a probabilistic approach.

In this paper, we continue the work from the papers mentioned above. The Theorem 2.1 is the general result which will be applied for particular cases of operators. The contents and the proof of Theorem 2.1 a) are the same as those of Theorem 2 from [5], but for a larger class of operators. The part b) of the same theorem extends the result from [7]. But the main innovation, which is contained in the Remarks 2.1 and 2.2, is a new method to find an appropriate function φ in connection with the Theorem 2.1. The results obtained in Corollary 3.1, 3.2 and 3.3 are not new, see [7] and [5], but are included to present our simpler technique. The first part of Corollary 3.4 is also known, see [6], however the second part of this corollary and the Corollary 3.5 are new results.

2 Main result

Let $\varphi\colon I \to J$ be a continuously differentiable one-to-one map. Define the space

$$UC_\varphi = \left\{\, f \in C(I) \mid f \circ \varphi^{-1} \text{ is uniformly continuous on } J \,\right\}.$$

For a function $f \in UC_\varphi$ we define the following modulus of continuity

$$\omega^\varphi(f,\delta) = \sup_{\substack{x,t\in I\\ |\varphi(x)-\varphi(t)|\le\delta}} |f(x)-f(t)|, \quad \text{for } \delta > 0,$$

which generalizes the usual modulus of continuity. Actually, we have the relation

$$\omega^\varphi(f,\delta) = \omega(f\circ\varphi^{-1},\delta).$$

Proposition 2.1. *For every function $f \in UC_\varphi$ we have*

$$\lim_{n\to\infty} \omega^\varphi(f,\delta_n) = 0 \textit{ whenever } \delta_n \to 0.$$

The converse is also true, i.e. if there is a positive sequence δ_n converging to 0, such that $\omega^\varphi(f,\delta_n)\to 0$, then $f\in UC_\varphi$.

Proof. This property is true due to the property of the usual modulus of continuity to be continuous at 0 only for an uniformly continuous function. □

Proposition 2.2. *For every function $f \in UC_\varphi$ and for every $t,x\in I$ and $\delta \ge 0$ we have*

$$|f(t)-f(x)| \le \left(1+\frac{|\varphi(t)-\varphi(x)|}{\delta}\right)\omega^\varphi(f,\delta).$$

Proof. Let $t,x\in I$. We have

$$|f(t)-f(x)| \le \omega^\varphi(f,|\varphi(t)-\varphi(x)|) \le \left(1+\frac{|\varphi(t)-\varphi(x)|}{\delta}\right)\omega^\varphi(f,\delta),$$

the last inequality being true because

$$\omega^\varphi(f,\lambda\delta) = \omega(f\circ\varphi^{-1},\lambda\delta) \le (1+\lambda)\cdot\omega(f\circ\varphi^{-1},\delta) = (1+\lambda)\cdot\omega^\varphi(f,\delta).$$

□

Theorem 2.1. *Let $A_n\colon C(I)\to C(I)$ be a sequence of positive linear operators preserving constant functions. Then*
a) if $\sup_{x\in I} A_n(|\varphi(t)-\varphi(x)|,x) = a_n \to 0$ and $f\in UC_\varphi$, then $\|A_nf-f\|_\infty \to 0$ and moreover

$$\|A_nf-f\|_\infty \le 2\cdot\omega^\varphi(f,a_n).$$

b) if $\|A_nf-f\|_\infty\to 0$ and $A_nf\in UC_\varphi$, then $f\in UC_\varphi$.

Proof. Applying the positive operators A_n to the inequality

$$|f(t) - f(x)| \leq \left(1 + \frac{|\varphi(t) - \varphi(x)|}{\delta}\right) \omega^{\varphi}(f, \delta),$$

we obtain

$$|A_n(f, x) - f(x)| \leq \left(1 + \frac{A_n|\varphi(t) - \varphi(x)|}{\delta_n}\right) \cdot \omega^{\varphi}(f, \delta_n).$$

Choosing $\delta_n = a_n$, we obtain the inequality

$$\|A_n f - f\|_{\infty} \leq 2 \cdot \omega^{\varphi}(f, a_n).$$

Using the result from Proposition 2.1, we obtain a). To prove b), we use the fact that ω^{φ} is a seminorm and obtain

$$\omega^{\varphi}(f, \delta_n) \leq \omega^{\varphi}(f - A_n f, \delta_n) + \omega^{\varphi}(A_n f, \delta_n) \leq 2\,\|f - A_n f\|_{\infty} + \omega^{\varphi}(A_n f, \delta_n).$$

Using the result from Proposition 2.2, we obtain $\omega^{\varphi}(A_n f, \delta_n) \to 0$, for a sequence $\delta_n \to 0$. We obtain that $\omega^{\varphi}(f, \delta_n) \to 0$, which proves that $f \in UC_{\varphi}$. □

Remark 2.1. *For the positive linear operators A_n that transform the continuous functions into differentiable functions we have the following method to find the function φ and to prove that $A_n f$ is from UC_{φ}: by the Cauchy mean-value formula we have for some c between x and t, the relation*

$$\varphi'(c)[A_n f(x) - A_n f(t)] = (A_n f)'(c)[\varphi(x) - \varphi(t)],$$

so

$$\omega^{\varphi}(A_n f, \delta_n) = \sup_{\substack{x,t \in I \\ |\varphi(x) - \varphi(t)| \leq \delta_n}} |A_n f(x) - A_n f(t)| \leq \delta_n \cdot \sup_{c \in I} \left|\frac{(A_n f)'(c)}{\varphi'(c)}\right|.$$

If the quantity $\|(A_n f)'/\varphi'\|_{\infty}$ is finite for all n, then we obtain the desired convergence $\omega^{\varphi}(A_n f, \delta_n) \to 0$, as $\delta_n \to 0$.

Remark 2.2. *Now, having found a function ϕ for which $\|(A_n f)'/\phi'\|_{\infty}$ is finite for every n, it remains to prove that $\sup_{x \in I} A_n(|\phi(x) - \phi(t)|, x)$ is a sequence converging to 0. But, in most of the cases, this is very difficult. To overcome this, we use an idea from [7]: we take a function φ for which we have $\sup_{x \in I} A_n(|\varphi(x) - \varphi(t)|, x) \to 0$, when $n \to \infty$, and prove that f belongs to UC_{ϕ} if and only if f belongs to UC_{φ}. This equivalence is true if we prove that $\phi \circ \varphi^{-1}$ and $\varphi \circ \phi^{-1}$ are uniform continuous functions.*

3 Applications

Corollary 3.1. *For the Szász-Mirakjan operators $S_n \colon C[0, \infty) \to C[0, \infty)$ defined by*

$$S_n f(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),$$

we have $||S_n f - f||_\infty \to 0$ *if* $f(x^2)$ *is uniformly continuous on* $[0,\infty)$*. If* f *is bounded and continuous on* $[0,\infty)$ *and* $||S_n f - f||_\infty \to 0$ *then* $f(x^2)$ *is uniformly continuous on* $[0,\infty)$*. We have also the estimation*

$$||S_n f - f||_\infty \leq 2 \cdot \omega \left(f(t^2), \frac{1}{\sqrt{n}} \right), \quad n \geq 1.$$

Proof. We have (see for example [2]) $S_n(1,x) = 1, \ \ S_n(t,x) = x, \ \ S_n(t^2,x) = x^2 + \dfrac{x}{n}$. Using the Cauchy-Schwarz inequality for positive linear functionals we have

$$S_n(|t-x|,x) \leq \sqrt{S_n((t-x)^2,x)} = \sqrt{S_n(t^2,x) - x^2} = \sqrt{\frac{x}{n}}.$$

For a bounded function $f \in C[0,\infty)$ the derivative $(S_n f)'(x)$ fulfills

$$\begin{aligned} |(S_n f)'(x)| &= \left| \frac{n}{x} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \left(x - \frac{k}{n}\right) e^{-nx} \frac{(nx)^k}{k!} \right| \\ &\leq \frac{n}{x} \|f\|_\infty S_n(|t-x|,x) \leq \|f\|_\infty \frac{\sqrt{n}}{\sqrt{x}}. \end{aligned}$$

We choose the function $\varphi(x) = \sqrt{x}$ and $\delta_n = 1/n$, and using the Remark 2.1 we obtain

$$\omega^\varphi(S_n f, \delta_n) \leq \delta_n \cdot \sup_{x \in I} \left| \frac{(S_n f)'(x)}{\varphi'(x)} \right| \leq \frac{2||f||_\infty}{\sqrt{n}},$$

which proves that $S_n f \circ \varphi^{-1}$ is uniformly continuous. Using Theorem 2.1 b) we deduce that $f \circ \varphi^{-1}(x) = f(x^2)$ is uniformly continuous if $S_n f$ converges uniformly to f on $[0,\infty)$. To prove the other part of the Corollary, we use

$$|\varphi(x) - \varphi(t)| = \frac{|x-t|}{\sqrt{x} + \sqrt{t}} \leq \frac{|x-t|}{\sqrt{x}}, \quad \text{for } x > 0, \ t \geq 0$$

to obtain

$$a_n = \sup_{x>0} S_n(|\varphi(x) - \varphi(t)|, x) \leq \sup_{x>0} \frac{S_n(|t-x|,x)}{\sqrt{x}} \leq \frac{1}{\sqrt{n}}.$$

By Theorem 2.1 a) we obtain that $S_n f$ converges uniformly to f on $[0,\infty)$, if $f(x^2)$ is uniformly continuous on $[0,\infty)$. □

Corollary 3.2. *For the Baskakov operators* $V_n \colon C[0,\infty) \to C[0,\infty)$ *defined by*

$$V_n f(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right),$$

we have $||V_n f - f||_\infty \to 0$*, if* $f(e^x - 1)$ *is uniformly continuous on* $[0,\infty)$*. If* f *is bounded and continuous on* $[0,\infty)$ *and* $V_n f$ *converges uniformly to* f *on* $[0,\infty)$*, then* $f(e^x - 1)$ *is uniformly continuous on* $[0,\infty)$*. We have also the estimation*

$$||V_n f - f||_\infty \leq 2 \cdot \omega \left(f(e^t - 1), \frac{1}{\sqrt{n-1}} \right), \quad n \geq 2.$$

Proof. We have (see [2]) $V_n(1,x)=1,\ \ V_n(t,x)=x,\ \ V_n(t^2,x)=x^2+\dfrac{x(1+x)}{n}$. Using the Cauchy-Schwarz inequality we have

$$V_n(|t-x|,x) \le \sqrt{V_n((t-x)^2,x)} = \sqrt{V_n(t^2,x)-x^2} = \sqrt{\frac{x(1+x)}{n}}.$$

For a bounded function $f \in C[0,\infty)$, computing the derivative

$$\begin{aligned} |(V_n f)'(x)| &= \left| \frac{n}{x(1+x)} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\left(x-\frac{k}{n}\right)\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}} \right| \\ &\le \frac{n}{x(1+x)} \|f\|_\infty V_n(|t-x|,x) \le \|f\|_\infty \frac{\sqrt{n}}{\sqrt{x(1+x)}}, \end{aligned}$$

we can choose the function $\phi(x) = \ln\left(x+\frac{1}{2}+\sqrt{x(1+x)}\right)$ and $\delta_n = 1/n$. Because $\phi'(x) = 1/\sqrt{x(1+x)}$, using the Remark 2.1, we obtain

$$\omega^\phi(V_n f, \delta_n) \le \delta_n \cdot \sup_{x\in I} \left| \frac{(V_n f)'(x)}{\phi'(x)} \right| \le \frac{||f||_\infty}{\sqrt{n}},$$

which proves that $S_n f \circ \phi^{-1}$ is uniformly continuous. Using Theorem 2.1 b) we deduce that $(f\circ\phi^{-1})(x) = f\left((2e^x-1)^2/8e^x\right)$ is uniformly continuous if $V_n f$ converges uniformly to f on $[0,\infty)$. But the uniform continuity of $f\left((2e^x-1)^2/8e^x\right)$ is equivalent to the uniform continuity of $f(e^x-1)$. Indeed, let $\varphi(x) = \ln(1+x)$. Because the function

$$(\varphi\circ\phi^{-1})(x) = \ln\left(1+\frac{(2e^x-1)^2}{8e^x}\right) = 2\ln(2e^x+1) - x - \ln 8$$

is uniformly continuous on $[0,\infty)$ (having a bounded derivative) and because

$$(\phi\circ\varphi^{-1})(x) = \ln\left(e^x - \frac{1}{2} + \sqrt{(e^x-1)e^x}\right)$$

is also uniformly continuous on $[0,\infty)$ (being a continuous function on $[0,\infty)$ with the property that $(\phi\circ\varphi^{-1})(x) - x$ has a finite limit at infinity), we have proved the equivalence stated. To prove the other part, we use the Geometric, Logarithmic, Arithmetic Mean Inequality

$$\sqrt{uv} < \frac{u-v}{\ln u - \ln v} < \frac{u+v}{2}, \quad \text{true for every } 0<v<u.$$

We obtain

$$|\varphi(x)-\varphi(t)| = |\ln(1+x)-\ln(1+t)| \le \frac{|x-t|}{\sqrt{(1+x)(1+t)}} = \left| \sqrt{\frac{1+x}{1+t}} - \sqrt{\frac{1+t}{1+x}} \right|.$$

Using the Cauchy-Schwarz inequality we get

$$\begin{aligned} a_n &= \sup_{x\geq 0} V_n(|\varphi(x)-\varphi(t)|,x) \\ &\leq \sup_{x\geq 0}\sqrt{(1+x)V_n\left(\frac{1}{1+t},x\right)+\frac{V_n(1+t,x)}{1+x}-2V_n(1,x)} \end{aligned}$$

and because

$$\begin{aligned} V_n\left(\frac{1}{1+t},x\right) &= \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}\cdot\frac{n}{n+k} \\ &\leq \frac{n}{(n-1)(1+x)}\sum_{k=0}^{\infty}\binom{n+k-2}{k}\frac{x^k}{(1+x)^{n-1+k}} \\ &= \frac{n}{(n-1)(1+x)} \end{aligned}$$

we obtain

$$a_n \leq \sqrt{\frac{n}{n-1}+1-2} = \frac{1}{\sqrt{n-1}}, \quad \text{for } n\geq 2.$$

By Theorem 2.1 a) we obtain that $V_n f$ converges uniformly to f on $[0,\infty)$, if $f(e^x-1)$ is uniformly continuous on $[0,\infty)$. □

Corollary 3.3. *For the Meyer-König and Zeller operators $M_n\colon C[0,1)\to C[0,1)$ defined by*

$$M_n(f,x)=\sum_{k=0}^{\infty}\binom{n+k}{k}x^k(1-x)^{n+1}f\left(\frac{k}{n+k}\right)$$

we have $||M_n f-f||_\infty\to 0$, for those f such that $f(1-e^{-x})$ is uniformly continuous on $[0,\infty)$. If f is bounded and continuous on $[0,1)$ and $M_n f$ converges uniformly on $[0,1)$ to f, then $f(1-e^{-x})$ is uniformly continuous on $[0,\infty)$. We have also the estimation

$$||M_n f-f||_\infty \leq 2\cdot\omega\left(f(1-e^{-t}),\frac{1}{\sqrt{n}}\right), \quad \textit{for } n\geq 1.$$

Proof. We have (see [3]) $M_n(1,x)=1$, $M_n(t,x)=x$ and

$$\frac{x(1-x)^2}{n+1}\leq M_n(t^2,x)-x^2\leq\frac{2x(1-x)^2}{n+1}, \quad \text{for } n\geq 3.$$

From these relations we obtain

$$M_n(|t-x|,x)\leq\sqrt{M_n((t-x)^2,x)}=\sqrt{M_n(t^2,x)-x^2}\leq\frac{(1-x)\sqrt{2x}}{\sqrt{n+1}}.$$

A simple computation yields

$$\begin{aligned}
&|(M_n f)'(x)| \\
&= \left| \frac{n+1}{x(1-x)^2} \sum_{k=0}^{\infty} \binom{n+k+1}{k} x^k (1-x)^{n+2} \left(\frac{k}{n+1+k} - x \right) f\left(\frac{k}{n+k} \right) \right| \\
&\le \frac{n+1}{x(1-x)^2} \cdot ||f||_\infty \cdot M_{n+1}(|t-x|, x) \le ||f||_\infty \frac{\sqrt{2(n+1)}}{(1-x)\sqrt{x}}.
\end{aligned}$$

We choose $\phi(x) = \ln \frac{1+\sqrt{x}}{1-\sqrt{x}}$, because $\phi'(x) = \frac{1}{(1-x)\sqrt{x}}$. We have $\phi^{-1}(x) = \left(\frac{e^x-1}{e^x+1}\right)^2$, so using Remark 2.1 and Theorem 2.1 b), $f \circ \phi^{-1}$ is uniformly continuous on $[0, \infty)$, if we have $||M_n f - f||_\infty \to 0$. But the uniform continuity of $f \circ \phi^{-1}$ on $[0, \infty)$ is equivalent with the uniform continuity of the function $f(1 - e^{-x})$ on $[0, \infty)$. Indeed, let $\varphi(x) = -\ln(1-x)$. If we prove that the functions $\phi \circ \varphi^{-1}$ and $\varphi \circ \phi^{-1}$ are uniformly continuous, the equivalence is proved. But

$$(\phi \circ \varphi^{-1})(x) - x = \ln \frac{1+\sqrt{1-e^{-x}}}{(1-\sqrt{1-e^{-x}})e^x} = 2\ln\left(1+\sqrt{1-e^{-x}}\right)$$

is a continuous function on $[0, \infty)$ having a finite limit at infinity, so, $\phi \circ \varphi^{-1}$ is uniformly continuous on $[0, \infty)$. The function

$$(\varphi \circ \phi^{-1})(x) = -\ln\left(1 - \left(\frac{e^x-1}{e^x+1}\right)^2\right) = 2\ln(e^x+1) - x - \ln 4$$

has a bounded derivative, so it is uniformly continuous on $[0, \infty)$.
To prove the other part, we use the same Geometric, Logarithmic, Arithmetic Mean Inequality and we obtain

$$|\varphi(x) - \varphi(t)| = |-\ln(1-x) + \ln(1-t)| \le \frac{|x-t|}{\sqrt{(1-x)(1-t)}} = \left| \frac{\sqrt{1-x}}{\sqrt{1-t}} - \frac{\sqrt{1-t}}{\sqrt{1-x}} \right|.$$

We have

$$a_n = \sup_{x \in [0,1)} M_n(|\varphi(x) - \varphi(t)|, x) \le \sup_{x \in [0,1)} \sqrt{1 - x + \frac{n+1}{n} x + 1 - 2} = \frac{1}{\sqrt{n}},$$

because

$$M_n\left(\frac{1}{1-t}, x\right) = \sum_{k=0}^{\infty} \binom{n+k}{k} x^k (1-y)^{n+1} \cdot \frac{n+k}{n} = 1 + \frac{n+1}{n} \cdot \frac{x}{1-x}.$$

By Theorem 2.1 a) we obtain that $M_n f$ converges uniformly to f on $[0, 1)$, if the function $f(1 - e^{-x})$ is uniformly continuous on $[0, \infty)$. □

Corollary 3.4. *For the Gauss-Weierstrass operators*

$$W_n(f,x) = \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-n\frac{(u-x)^2}{2}} f(u)\,du,$$

defined for continuous functions on $\mathbb{R}$, for which the integral is finite, we have the uniform convergence $||W_n f - f||_\infty \to 0$, if f is uniformly continuous on $\mathbb{R}$. If f is bounded and continuous on $\mathbb{R}$ and $P_n f$ converges uniformly to f on the whole real axis, then f is uniformly continuous on $\mathbb{R}$. We have also the estimation

$$||W_n f - f||_\infty \leq 2\cdot\omega\left(f, \frac{1}{\sqrt{n}}\right), \quad \textit{for } n \geq 1.$$

Proof. We have $W_n(1,x) = 1$, $W_n(t,x) = x$, $W_n(t^2,x) = x^2 + \frac{1}{n}$, and

$$W_n(|t-x|,x) \leq \sqrt{W_n((t-x)^2,x)} = \sqrt{W_n(t^2,x) - x^2} \leq \frac{1}{\sqrt{n}}.$$

Because the derivative

$$|(W_n f)'(x)| = n|W_n((t-x)f(t),x)| \leq n||f||_\infty W_n(|t-x|,x) \leq \sqrt{n}||f||_\infty,$$

is bounded for every n, for a bounded and continuous function f, we consider $\varphi(x) = x$, and by Remark 2.1 and Theorem 2.1 b), we obtain that f is uniformly continuous, if $||W_n f - f||_\infty \to 0$. Conversely, $W_n f$ converges to f uniformly on $\mathbb{R}$, if f is uniformly continuous on $\mathbb{R}$, because of the Theorem 2.1 a) and because the sequence

$$a_n = \sup_{x\in\mathbb{R}} W_n(|\varphi(x) - \varphi(t)|,x) = \sup_{x\in\mathbb{R}} W_n(|x-t|,x) \leq \frac{1}{\sqrt{n}}$$

converges to 0. □

Corollary 3.5. *For the Bleimann-Butzer-Hahn operators $L_n\colon C[0,\infty) \to C[0,\infty)$ defined by*

$$L_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1+x)^{-n} f\left(\frac{k}{n-k+1}\right)$$

we have $||L_n f - f||_\infty \to 0$, for those f such that $f(x^{-2}-1)$ is uniformly continuous on $(0,1]$. If f is bounded and continuous on $[0,\infty)$ and $L_n f$ converges uniformly on $[0,\infty)$ to f, then $f(x^{-2}-1)$ is uniformly continuous on $(0,1]$. We have also the estimation

$$||L_n f - f||_\infty \leq 2\cdot\omega\left(f(t^{-2}-1), \frac{1}{\sqrt{n+1}}\right), \quad \textit{for } n \geq 1.$$

Proof. We have (see [4]) $M_n(1,x) = 1$, $M_n(t,x) = x - x\left(\frac{x}{1+x}\right)^n$ and (see [1])

$$L_n((t-x)^2,x) \leq \frac{3x(1+x)^2}{n+2}, \quad \text{for } n \geq 1 \text{ and } x \geq 0.$$

From these relations we obtain

$$L_n(|t-x|,x) \le \sqrt{L_n((t-x)^2,x)} \le \frac{(1+x)\sqrt{3x}}{\sqrt{n+2}}.$$

A simple computation yields

$$\begin{aligned} |(L_nf)'(x)| &= \left| \frac{n+1}{x} \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{(1+x)^n} \left(\frac{k}{n+1} - \frac{nx}{(1+n)(1+x)} \right) f\left(\frac{k}{n-k+1}\right) \right| \\ &\le \frac{n+1}{x} \cdot ||f||_\infty \cdot L_n\left(\left| \frac{t}{1+t} - \frac{nx}{(1+n)(1+x)} \right|, x \right), \end{aligned}$$

and because

$$\begin{aligned} L_n\left(\frac{t}{1+t}, x\right) &= \frac{nx}{(1+n)(1+x)} \\ L_n\left(\left(\frac{t}{1+t}\right)^2, x\right) &= \frac{n^2x^2}{(1+n)^2(1+x)^2} + \frac{nx}{(1+n)^2(1+x)^2} \end{aligned}$$

we obtain, by applying the Cauchy-Schwarz inequality

$$|(L_nf)'(x)| \le ||f||_\infty \frac{\sqrt{n}}{(1+x)\sqrt{x}}.$$

We choose $\phi(x) = 2 \arctan\sqrt{x}$, because $\phi'(x) = \frac{1}{(1+x)\sqrt{x}}$. The inverse of ϕ, namely $\phi^{-1}(x) = \left(\tan\frac{x}{2}\right)^2$ is defined for $x \in [0,\pi)$, so using Remark 2.1 and Theorem 2.1 b), $f \circ \phi^{-1}$ is uniformly continuous on $[0,\pi)$, if we have $||L_nf - f||_\infty \to 0$. But the uniform continuity of $f \circ \phi^{-1}$ on $[0,\pi)$ is equivalent with the uniform continuity of the function $f(x^{-2}-1)$ on $(0,1]$. Indeed, let $\varphi(x) = \frac{1}{\sqrt{1+x}}$. If we prove that the functions $\phi \circ \varphi^{-1}$ and $\varphi \circ \phi^{-1}$ are uniformly continuous, the equivalence is proved. But

$$(\phi \circ \varphi^{-1})(x) = 2 \arctan \frac{\sqrt{1-x^2}}{x}$$

is a continuous function on $(0,1]$ having a finite limit at $x=0$, so, $\phi \circ \varphi^{-1}$ is uniformly continuous on $(0,1]$. The function

$$(\varphi \circ \phi^{-1})(x) = \frac{1}{\sqrt{1+\left(\tan\frac{x}{2}\right)^2}} = \cos\frac{x}{2}$$

has a bounded derivative, so it is uniformly continuous on $[0,\pi)$.
To prove the other part, we use the inequality

$$|\varphi(x) - \varphi(t)| = \left| \frac{1}{\sqrt{1+x}} - \frac{1}{\sqrt{1+t}} \right| = \frac{\left|\sqrt{1+t} - \sqrt{1+x}\right|}{\sqrt{(1+x)(1+t)}} \le \frac{1}{\sqrt{1+x}} \frac{|t-x|}{\sqrt{(1+x)(1+t)}}$$

and applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} L_n(|\varphi(x)-\varphi(t)|,x) &\le \frac{1}{\sqrt{1+x}}\sqrt{L_n\left(\frac{(x-t)^2}{(1+x)(1+t)},x\right)} \\ &= \frac{1}{\sqrt{1+x}}\sqrt{L_n\left(\frac{1+x}{1+t},x\right)+L_n\left(\frac{1+t}{1+x},x\right)-2\,L_n(1,x)}. \end{aligned}$$

Using

$$L_n\left(\frac{1+x}{1+t},x\right) = (1+x)L_n\left(1-\frac{t}{1+t},x\right) = 1+x-\frac{nx}{n+1} = 1+\frac{x}{n+1}$$

and

$$L_n\left(\frac{1+t}{1+x},x\right) = \frac{1}{1+x}L_n\left(1+t,x\right) = \frac{1+x-x\left(\frac{x}{1+x}\right)^n}{1+x} = 1-\left(\frac{x}{1+x}\right)^{n+1}$$

we deduce that the sequence

$$a_n = \sup_{x\ge 0} L_n(|\varphi(x)-\varphi(t)|,x) \le \sup_{x\ge 0}\frac{1}{\sqrt{1+x}}\sqrt{\frac{x}{n+1}-\left(\frac{x}{1+x}\right)^{n+1}} \le \frac{1}{\sqrt{n+1}}$$

is convergent to 0, so, by Theorem 2.1 a) the sequence L_nf converges uniformly to f on $[0,\infty)$, if $f(x^{-2}-1)$ is uniformly continuous on $(0,1]$. □

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