The rate of approximation of real functions by rational functions with prescribed numerator degree

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Abstract

We give estimations of the approximation of positive real functions by reciprocals of polynomials and of approximation of functions that change sign by rational functions with prescribed numerator degree, in terms of first order modulus of smoothness of Ditzian and Totik.

1 Introduction

In [6], the authors show that one can approximate a nonconstant positive real function by reciprocals of real polynomials at the rate $\omega(f, 1/n)$, where $\omega(f, \cdot)$ is the usual modulus of continuity. In [5], the rate is given by $\omega_1^\varphi(f, 1/n)$, the first order modulus of Ditzian and Totik. Our Lemma 3.1 and Theorem 3.1 give a new proof of the result obtained in [5].

Using the technique from [7, Theorem 2.1], we derive in Lemma 3.2 a better inequality with respect to the order of n. The inequality from Lemma 3.3 is given in [4, Theorem 3.1], but we give a simpler proof. Theorem 3.2 is the result from [4] and is presented for the sake of completeness.

2 Preliminaries

The Ditzian-Totik modulus of first order is defined by

$$\omega_1^\varphi(f,\delta) = \sup_{|h|\le\delta} \sup_{x\pm(h/2)\varphi(x)\in[0,1]} \left| f\left(x+\frac{h}{2}\varphi(x)\right) - f\left(x-\frac{h}{2}\varphi(x)\right)\right|, \quad \delta\ge 0,$$

for the step-weight $\varphi(x)=\sqrt{x(1-x)}$ and for a continuous function $f\in C[0,1]$. The K-functional related to this modulus is defined by

$$K_1^\varphi(f,\delta) = \inf_{g\in AC[0,1]} (\|f-g\| + \delta\,\|\varphi g'\|), \quad \delta\ge 0,$$

in which $\|\cdot\|$ denotes the uniform norm on $[0,1]$ and $AC[0,1]$ is the space of absolutely continuous functions defined on $[0,1]$. It is well-known (see [2]), that the K-functional $K_1^{\varphi}(f,\delta)$ and the modulus $\omega_1^{\varphi}(f,\delta)$ are equivalent.

The second order modulus of Ditzian-Totik for the step-weight $\varphi(x) = \sqrt{x(1-x)}$ and for $f \in C[0,1]$, is defined by

$$\omega_2^{\varphi}(f,\delta) = \sup_{|h|\le\delta} \sup_{x, x\pm h\varphi(x)\in[0,1]} |f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|, \quad \delta \ge 0.$$

It is well-known that this modulus is equivalent with the K-functional

$$K_2^{\varphi}(f,\delta) = \inf_{g'\in AC[0,1]} (\|f-g\| + \delta^2 \|\varphi^2 g''\|), \quad \delta \ge 0.$$

Consider the sequence of positive linear operators $A_n\colon C[0,1] \to \Pi_n$, where Π_n is the space of all polynomial with degree at most n. Suppose A_n has the properties

1. $A_n(e_i,x) = e_i(x), \quad i = 0,1, \text{ where } e_i(x) = x^i,$
2. $A_n((t-x)^2,x) \le C \cdot \dfrac{\varphi^2(x)}{n^2},$
3. $\|A_n f - f\| \le C \cdot \omega_1^{\varphi}\left(f, \dfrac{1}{n}\right),$
4. $A_n(f,x) \ge f(x), \quad 0 < x < 1,$ for every convex function f on $(0,1)$,
5. $|A_n(f,x) - f(x)| \le C\omega_2\left(f, \dfrac{\varphi(x)}{n}\right),$

In fact, properties 3,4 and 5 can be obtained from 1 and 2. An example of such operators are $A_n = H_{2\left[\frac{n-1}{2}\right]+1}\colon C[0,1] \to \Pi_n$, $n \ge 3$, which were defined in [3] in the following manner: For $n \ge 1$ let x_n be the greatest root of the Jacobi polynomial $J_n^{(1,0)}$ of degree n related to the interval $[0,1]$ and

$$P_{2n-1}(x) = \lambda_n \int_0^x \left(\frac{J_n^{(1,0)}(t)}{t-x_n}\right)^2 dt, \quad \text{where } \lambda_n = \frac{1}{\int_0^1 (1-x)\left(\frac{J_n^{(1,0)}(x)}{x-x_n}\right)^2 dx}.$$

If $P_{2n-1}(x) = \sum_{k=0}^{2n-1} a_k x^k$, then the operators $H_{2n+1}\colon C[0,1] \to \Pi_{2n+1}$ are defined by

$$H_{2n+1} f = \sum_{k=0}^{2n-1} \frac{a_k}{k+1} L_{k+2} f,$$

where the operators $L_n\colon C[0,1] \to \Pi_n$ are defined by

$$L_n(f,x) = f(0)(1-x)^n + f(1)x^n + (n-1)\sum_{k=1}^{n-1} p_{n,k}(x) \int_0^1 p_{n-2,k-1}(t) \cdot f(t)\, dt,$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

The operators H_{2n+1} are linear and positive, preserve the affine functions and

$$H_{2n+1}(e_2, x) - x^2 = x(1-x)\left(1 - \int_0^1 x^2 P_{2n-1}(x)\, dx\right) \leq \varphi^2(x)(1-x_n) \leq \frac{C\varphi^2(x)}{n^2}.$$

3 Main results

Lemma 3.1. *For the operators A_n we have the property*

$$A_n(|f(t) - f(x)|^2, x) \leq C \cdot \left[\omega_1^{\varphi}\left(f, \frac{1}{n}\right)\right]^2.$$

Proof. Because of the equivalence between $\omega_1^{\varphi}(f,t)$ and $K_1^{\varphi}(f,t)$, for each integer $n = 1, 2, \ldots$, there exists an absolutely continuous function f_n, such that

$$\|f - f_n\| \leq C_1 \omega_1^{\varphi}\left(f, \frac{1}{n}\right), \quad \text{and} \quad \|\varphi f_n'\| \leq C_2 n \omega_1^{\varphi}\left(f, \frac{1}{n}\right). \tag{3.1}$$

We have

$$|f_n(t) - f_n(x)| = \left|\int_x^t f_n'(u)\, du\right| \leq \|\varphi f_n'\| \left|\int_x^t \frac{du}{\varphi(u)}\right|.$$

Using the inequality (see [1])

$$\left|\int_x^t \frac{du}{\varphi(u)}\right| \leq 2 \cdot |t - x| \cdot \min\left(\frac{1}{\varphi(x)}, \frac{1}{\varphi(t)}\right),$$

the relations(3.1) and the properties of A_n, we obtain

$$A_n(|f_n(t) - f_n(x)|^2, x) \leq \|\varphi f_n'\|^2 A_n\left(\frac{4(t-x)^2}{\varphi^2(x)}, x\right) \leq C \cdot \left[\omega_1^{\varphi}\left(f, \frac{1}{n}\right)\right]^2.$$

Using the inequality $(a+b)^2 \leq 2(a^2 + b^2)$, we finally obtain

$$\begin{aligned} A_n(|f(t) - f(x)|^2, x) &= A_n(|f(t) - f_n(t) - (f(x) - f_n(x)) + f_n(t) - f_n(x)|^2, x) \\ &\leq 4 \cdot \|f - f_n\|^2 + A_n(|f_n(t) - f_n(x)|^2, x) \\ &\leq C \cdot \left[\omega_1^{\varphi}\left(f, \frac{1}{n}\right)\right]^2. \end{aligned}$$

□

Theorem 3.1. *Let $f \in C[0,1]$ be a nonconstant and nonnegative function. Then, there is a sequence of polynomials $p_n \in \Pi_n$ such that*

$$\left\|f - \frac{1}{p_n}\right\| \leq C \cdot \omega_1^{\varphi}\left(f, \frac{1}{n}\right).$$

Proof. For $\varepsilon > 0$, we define de function

$$f_\varepsilon(x) = f(x) + \varepsilon > 0, \quad x \in [0,1].$$

and consider the polynomials

$$p_n(x) = A_n\left(\frac{1}{f_\varepsilon}, x\right) > 0, \quad x \in [0,1].$$

Using Cauchy-Schwarz inequality for positive linear operators, we have

$$1 = [A_n(e_0, x)]^2 \leq A_n(f_\varepsilon, x) \cdot A_n\left(\frac{1}{f_\varepsilon}, x\right) = p_n(x) \cdot A_n(f_\varepsilon, x). \tag{3.2}$$

We define the set

$$E = \left\{ x \in [0,1] \mid p_n(x) < \frac{1}{f_\varepsilon(x)} \right\}.$$

For $x \in E$, we have by relation (3.2) and the properties of A_n

$$0 < \frac{1}{p_n(x)} - f_\varepsilon(x) \leq A_n(f_\varepsilon, x) - f_\varepsilon(x) \leq C\omega_1^\varphi\left(f_\varepsilon, \frac{1}{n}\right) = C\omega_1^\varphi\left(f, \frac{1}{n}\right). \tag{3.3}$$

For $x \notin E$, we have

$$\begin{aligned} 0 \leq f_\varepsilon(x) - \frac{1}{p_n(x)} &= \frac{f_\varepsilon(x)}{p_n(x)} A_n\left(\frac{1}{f_\varepsilon(t)}, x\right) - A_n\left(\frac{e_0(t)}{p_n(x)}, x\right) \\ &= A_n\left(\frac{f_\varepsilon(x)}{p_n(x)} \cdot \frac{[f_\varepsilon(x) - f_\varepsilon(t)]}{f_\varepsilon(t) \cdot f_\varepsilon(x)}, x\right) \\ &\leq A_n\left(\frac{f_\varepsilon(x)}{f_\varepsilon(t)} \cdot [f_\varepsilon(x) - f_\varepsilon(t)], x\right), \end{aligned}$$

because for $x \notin E$ we have $p_n(x) \cdot f_\varepsilon(x) \geq 1$. We deduce that

$$\begin{aligned} 0 \leq f_\varepsilon(x) - \frac{1}{p_n(x)} &\leq A_n\left([f_\varepsilon(x) - f_\varepsilon(t)], x\right) + A_n\left(\frac{[f_\varepsilon(t) - f_\varepsilon(x)]^2}{f_\varepsilon(t)}, x\right) \\ &\leq A_n\left([f(x) - f(t)], x\right) + \frac{1}{\varepsilon} A_n\left([f(t) - f(x)]^2, x\right), \end{aligned}$$

because $1/f_\varepsilon \leq 1/\varepsilon$. Taking $\varepsilon = \omega_1^\varphi(f, 1/n)$, which is not zero because f is nonconstant and using Lemma 3.1 we obtain for $x \notin E$

$$0 \leq f_\varepsilon(x) - \frac{1}{p_n(x)} \leq C\omega_1^\varphi\left(f, \frac{1}{n}\right).$$

From this and (3.3) we have $\|f_\varepsilon - 1/p_n\| \leq C\omega_1^\varphi(f, 1/n)$, so

$$\left\| f - \frac{1}{p_n} \right\| \leq \|f - f_\varepsilon\| + \left\| f_\varepsilon - \frac{1}{p_n} \right\| \leq C\omega_1^\varphi\left(f, \frac{1}{n}\right).$$

□

Lemma 3.2. *For any* $0 < b_1 < b_2 < \cdots < b_\ell < 1$, $\ell \geq 1$, *let*

$$\rho(x) = (x - b_1)(x - b_2) \cdots (x - b_\ell).$$

Then, there exists a polynomial $S_n \in \Pi_n$ *such that for any* $x \in [0, 1]$ *and* $n \geq \ell$ *we have*

$$0 \leq 1 - \frac{|\rho(x)|}{S_n(x)} \leq \min\left(1, \frac{C\ell}{n} \sum_{j=1}^{\ell} \frac{\varphi(x)}{|x - b_j|}\right).$$

Proof. Let $b \in (0, 1)$ and let $g_b(x) = |x - b|$. Using the property 4 of A_n for the nonnegative and convex function g_b, we have

$$A_n(g_b, x) \geq g_b(x) \geq 0, \quad 0 < x < 1.$$

Therefore,

$$0 \leq 1 - \frac{g_b(x)}{A_n(g_b, x)} \leq 1, \quad 0 < x < 1.$$

From $\omega_2(g_b, h) \leq 2h$, using the property 5 of A_n, we have

$$|A_n(g_b, x) - g_b(x)| \leq C\frac{\varphi(x)}{n}.$$

We obtain

$$0 \leq 1 - \frac{g_b(x)}{A_n(g_b, x)} = \frac{|A_n(g_b, x) - g_b(x)|}{A_n(g_b, x)} \leq \frac{C\varphi(x)}{n \cdot A_n(g_b, x)} \leq \frac{C\varphi(x)}{n \cdot g_b(x)}.$$

We deduce that for any $b \in (0, 1)$,

$$0 \leq 1 - \frac{|x - b|}{A_n(g_b, x)} \leq \min\left(1, \frac{C\varphi(x)}{n|x - b|}\right), \quad 0 < x < 1.$$

Because $A_n(g, 0) = g(0)$ and $A_n(g, 1) = g(1)$, the last inequality is valid also for $x = 0$ and $x = 1$. We define

$$S_n(x) = \prod_{j=1}^{\ell} A_{[n/\ell]}(g_{b_j}, x),$$

where $[x]$ denotes the greatest integer not exceeding x. S_n is a polynomial with the degree at most $\ell \cdot [n/\ell] \leq n$, having the properties

$$S_n(x) \geq \prod_{j=1}^{\ell} g_{b_j}(x) = |\rho(x)|$$

and

$$
\begin{aligned}
1-\frac{|\rho(x)|}{S_n(x)} &= 1-\prod_{j=1}^{\ell}\left(1-\left(1-\frac{|x-b_j|}{A_{[n/\ell]}(g_{b_j},x)}\right)\right)\\
&\le 1-\prod_{j=1}^{\ell}\left(1-\min\left(1,\frac{C\ell\varphi(x)}{n|x-b_j|}\right)\right)\\
&\le \sum_{j=1}^{\ell}\min\left(1,\frac{C\ell\varphi(x)}{n|x-b_j|}\right),
\end{aligned}
$$

where we have used the inequality (see [4])

$$
1-\prod_{j=1}^{\ell}(1-y_j)\le\sum_{j=1}^{\ell}y_j,\quad y_j\in[0,1].
$$

□

Lemma 3.3. *There exists an absolute constant $C>0$ such that for $t,x\in[0,1]$ and $f\in C[0,1]$,*

$$
|f(t)-f(x)|\cdot\min\left(1,\frac{\max(\varphi(t),\varphi(x))}{n|t-x|}\right)\le C\cdot\omega_1^{\varphi}\left(f,\frac{1}{n}\right).
$$

Proof. Using the relations from the proof of Lemma 3.1 we have

$$
\begin{aligned}
&|f(t)-f(x)|\cdot\min\left(1,\frac{\max(\varphi(t),\varphi(x))}{n|t-x|}\right)\\
&\qquad\le |f_n(t)-f_n(x)|\cdot\frac{\max(\varphi(t),\varphi(x))}{n|t-x|}+|f(t)-f_n(t)-[f(x)-f_n(x)]|\\
&\qquad\le \|\varphi f_n'\|\,2|t-x|\cdot\min\left(\frac{1}{\varphi(t)},\frac{1}{\varphi(x)}\right)\cdot\frac{\max(\varphi(t),\varphi(x))}{n|t-x|}+2\,\|f-f_n\|\\
&\qquad\le C\cdot\omega_1^{\varphi}\left(f,\frac{1}{n}\right).
\end{aligned}
$$

□

Theorem 3.2. *There exists an absolute constant $C>0$ with the property: if the function $f\in C[0,1]$ changes sign exactly $\ell\ge1$ times in $[0,1]$, say at $b_1,b_2,\dots,b_\ell$, then for each $n\ge 2\ell$, there exists a polynomial $p_n\in\Pi_n$, having the same sign as f in $(b_\ell,1)$ and such that for $x\in[0,1]$,*

$$
\left|f(x)-\frac{(x-b_1)(x-b_2)\cdots(x-b_\ell)}{p_n(x)}\right|\le C\ell^2\omega_1^{\varphi}\left(f,\frac{1}{n}\right).
$$

Proof. Let $0 < b_1 < b_2 \cdots < b_\ell < 1$ and $f \in C[0,1]$ change sign exactly at b_j, $1 \le j \le \ell$. We may assume that f is positive in $(b_\ell, 1)$. Let $\rho(x) = \prod_{j=1}^{\ell}(x-b_j)$. If p_n is a positive polynomial in $[0,1]$, then

$$\left| f(x) - \frac{\rho(x)}{p_n(x)} \right| = \left| |f(x)| - \frac{|\rho(x)|}{p_n(x)} \right|.$$

We set

$$p_n(x) = S_{[n/2]}(x) \cdot U_{[n/2]}(x) > 0,$$

where $S_{[n/2]}$ is the polynomial of degree $[n/2] \ge \ell$, from the Lemma 3.2, and $U_{[n/2]}$ is the polynomial from Theorem 3.1 with the property

$$\left| |f(x)| - \frac{1}{U_{[n/2]}(x)} \right| \le C \cdot \omega_1^\varphi \left(|f|, \frac{1}{n} \right) \le C \cdot \omega_1^\varphi \left(f, \frac{1}{n} \right).$$

Then

$$\begin{aligned}
\left| f(x) - \frac{\rho(x)}{p_n(x)} \right| &= \left| |f(x)| - \frac{|\rho(x)|}{p_n(x)} \right| \\
&= \left| |f(x)| \left(1 - \frac{|\rho(x)|}{S_{[n/2]}(x)} \right) + \frac{|\rho(x)|}{S_{[n/2]}(x)} \left(|f(x)| - \frac{1}{U_{[n/2]}(x)} \right) \right| \\
&\le |f(x)| \min \left(1, \frac{C\ell}{n} \sum_{j=1}^{\ell} \frac{\varphi(x)}{|x-b_j|} \right) + \left| |f(x)| - \frac{1}{U_{[n/2]}(x)} \right| \\
&\le C\ell \sum_{j=1}^{\ell} |f(x) - f(b_j)| \min \left(1, \frac{\varphi(x)}{n|x-b_j|} \right) + \omega_1^\varphi \left(f, \frac{1}{n} \right) \\
&\le C\ell^2 \omega_1^\varphi \left(f, \frac{1}{n} \right).
\end{aligned}$$

□

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