An integral formula of Green's type

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Abstract

We prove an integral formula of Green's type and we apply this result in the proof of a generalization of Stokes' integral formula.

1 Introduction

Stokes' Theorem is a well known theorem in multivariable calculus and has a long and interesting history (see the article [3]). In some recent theoretical problems of physics (see [1] and [2] and the references given there) there is an increasing need to apply this theorem in a context where the usual conditions of the theorem (the smoothness of the vector field and smoothness of the surface) no longer hold.

In this paper we prove an integral formula similar to Green's theorem (Lemma 2.1 and Lemma 2.2) and then we give the proof of the Stokes' theorem for vector fields which are allowed to be discontinuous in a finite number of points.

2 Main results

Lemma 2.1. *Let C be the boundary of the rectangle $D = [a, b] \times [c, d]$ oriented in the positive direction. Let $x \in C^1(D)$ and let $P : D \to \mathbb{R}$ be a bounded function which has only finitely many points of discontinuity in D and which has partial derivatives on D that are Riemann integrable on D. Then*

$$\int_C P \cdot \frac{\partial x}{\partial u} \,\mathrm{d}u + P \cdot \frac{\partial x}{\partial v} \,\mathrm{d}v = \iint_D \left(\frac{\partial P}{\partial u} \cdot \frac{\partial x}{\partial v} - \frac{\partial P}{\partial v} \cdot \frac{\partial x}{\partial u} \right) \mathrm{d}u \,\mathrm{d}v. \tag{2.1}$$

Remark 2.1. *If $x \in C^2(D)$ then relation (2.1) is a consequence of Green's*

Theorem.

$$\begin{aligned}
&\int_C P \cdot \frac{\partial x}{\partial u}\,\mathrm{d}u + P \cdot \frac{\partial x}{\partial v}\,\mathrm{d}v \\
&\qquad = \iint_D \left[\left(P \cdot \frac{\partial x}{\partial v} \right)'_u - \left(P \cdot \frac{\partial x}{\partial u} \right)'_v \right] \mathrm{d}u\,\mathrm{d}v \\
&\qquad = \iint_D \left(\frac{\partial P}{\partial u} \cdot \frac{\partial x}{\partial v} + P \cdot \frac{\partial^2 x}{\partial u \partial v} - \frac{\partial P}{\partial v} \cdot \frac{\partial x}{\partial u} - P \cdot \frac{\partial^2 x}{\partial v \partial u} \right) \mathrm{d}u\,\mathrm{d}v \\
&\qquad = \iint_D \left(\frac{\partial P}{\partial u} \cdot \frac{\partial x}{\partial v} - \frac{\partial P}{\partial v} \cdot \frac{\partial x}{\partial u} \right) \mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

Proof of Lemma 2.1. The conditions over P and x assure the existence of the integrals of relation (2.1). Moreover, the functions $\frac{\partial P}{\partial v}$ and $\frac{\partial P}{\partial u}$ being Riemann integrable on $[a, b] \times [c, d]$ they are absolutely integrable, too.

Let $B_{n,m}f$ be the bivariate Bernstein polynomials for the function f and for the rectangle D:

$$B_{n,m}f(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} f\left(a + i\,\frac{b-a}{n}, c + j\,\frac{d-c}{m}\right) p_{n,i}\left(\frac{u-a}{b-a}\right) p_{m,j}\left(\frac{v-c}{d-c}\right),$$

where $p_{n,i}(x) = \binom{n}{i} x^k (1-x)^{n-k}$.

Using the Remark 2.1 for $B_{n,m}x \in C^2(D)$ we have

$$\int_C P \cdot \frac{\partial B_{n,m}x}{\partial u}\,\mathrm{d}u + P \cdot \frac{\partial B_{n,m}x}{\partial v}\,\mathrm{d}v = \iint_D \left(\frac{\partial P}{\partial u} \cdot \frac{\partial B_{n,m}x}{\partial v} - \frac{\partial P}{\partial v} \cdot \frac{\partial B_{n,m}x}{\partial u} \right) \mathrm{d}u\,\mathrm{d}v.$$

It is known (see for example [4]) that for every $f \in C^1(D)$ we have

$$\frac{\partial B_{n,m}f}{\partial u} \to \frac{\partial f}{\partial u} \text{ and } \frac{\partial B_{n,m}f}{\partial v} \to \frac{\partial f}{\partial v},$$

uniformly on D when $n, m \to \infty$.

Let $\varepsilon > 0$ be an arbitrary real number. For sufficiently large n and m we have

$$\begin{aligned}
&\left| \int_C P \cdot \frac{\partial x}{\partial u}\,\mathrm{d}u + P \cdot \frac{\partial x}{\partial v}\,\mathrm{d}v - \int_C P \cdot \frac{\partial B_{n,m}x}{\partial u}\,\mathrm{d}u + P \cdot \frac{\partial B_{n,m}x}{\partial v}\,\mathrm{d}v \right| \\
&\qquad = \left| \int_C P \cdot \left(\frac{\partial x}{\partial u} - \frac{\partial B_{n,m}x}{\partial u} \right) \mathrm{d}u + P \cdot \left(\frac{\partial x}{\partial v} - \frac{\partial B_{n,m}x}{\partial v} \right) \mathrm{d}v \right| \\
&\qquad \le \varepsilon \cdot \sup_{(u,v) \in D} |P(u, v)| \cdot 2(b - a + d - c),
\end{aligned}$$

and

$$\begin{aligned}
&\left| \iint_D \left(\frac{\partial P}{\partial u} \cdot \frac{\partial x}{\partial v} - \frac{\partial P}{\partial v} \cdot \frac{\partial x}{\partial u} \right) \mathrm{d}u\,\mathrm{d}v - \iint_D \left(\frac{\partial P}{\partial u} \cdot \frac{\partial B_{n,m}x}{\partial v} - \frac{\partial P}{\partial v} \cdot \frac{\partial B_{n,m}x}{\partial u} \right) \mathrm{d}u\,\mathrm{d}v \right| \\
&\qquad \le \iint_D \left(\left| \frac{\partial P}{\partial u} \right| \cdot \left| \frac{\partial x}{\partial v} - \frac{\partial B_{n,m}x}{\partial v} \right| + \left| \frac{\partial P}{\partial v} \right| \cdot \left| \frac{\partial x}{\partial u} - \frac{\partial B_{n,m}x}{\partial u} \right| \right) \mathrm{d}u\,\mathrm{d}v \\
&\qquad \le \varepsilon \cdot \iint_D \left(\left| \frac{\partial P}{\partial u}(u, v) \right| + \left| \frac{\partial P}{\partial v}(u, v) \right| \right) \mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

We obtain

$$\left|\int_C P\cdot\frac{\partial x}{\partial u}\,\mathrm{d}u + P\cdot\frac{\partial x}{\partial v}\,\mathrm{d}v - \iint_D\left(\frac{\partial P}{\partial u}\cdot\frac{\partial x}{\partial v} - \frac{\partial P}{\partial v}\cdot\frac{\partial x}{\partial u}\right)\mathrm{d}u\,\mathrm{d}v\right| \le \varepsilon\cdot C,$$

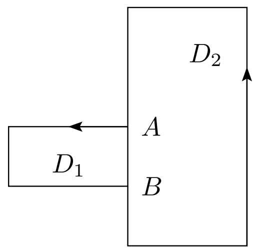
where C is the real number defined by

$$C = \sup_{(u,v)\in D}|P(u,v)|\cdot 2(b-a+d-c) + \iint_D\left(\left|\frac{\partial P}{\partial u}(u,v)\right| + \left|\frac{\partial P}{\partial v}(u,v)\right|\right)\mathrm{d}u\,\mathrm{d}v.$$

Letting $\varepsilon \searrow 0$ we obtain relation (2.1). □

Remark 2.2. *The relation (2.1) remains true if D is a simply connected domain formed by joining a finite number of rectangles in such a way that they have disjoint interiors and such that each rectangle has at least two points in common with another rectangle.*

It is sufficient to prove this for two rectangles D_1 and D_2 which have a common boundary on the segment AB. Let C_1 and C_2 be the boundaries of D_1



and D_2 oriented positively (traversed in the direction such that the interior of the rectangle is "to the left"). On C_1 the segment is traversed from B to A and on C_2 from A to B. Denote by D the union of points $D_1 \cup D_2$ and by C the boundary of D oriented in the positive direction. The double integral on D is the sum of double integrals on D_1 and on D_2. The line integral along C is the sum of line integrals along C_1 and C_2 because the line integral on AB cancels the line integral on BA.

Lemma 2.2. *Let D be a closed bounded simply connected domain in $\mathbb{R}^2$, which is bounded by the simple closed positively oriented piecewise smooth curve C. Let $P : D \to \mathbb{R}$ be a bounded function which has only finitely many points of discontinuity in D and which has partial derivatives which are Riemann integrable on D. If $x \in C^1(D)$, then*

$$\int_C P\cdot\frac{\partial x}{\partial u}\,\mathrm{d}u + P\cdot\frac{\partial x}{\partial v}\,\mathrm{d}v = \iint_D\left(\frac{\partial P}{\partial u}\cdot\frac{\partial x}{\partial v} - \frac{\partial P}{\partial v}\cdot\frac{\partial x}{\partial u}\right)\mathrm{d}u\,\mathrm{d}v.$$

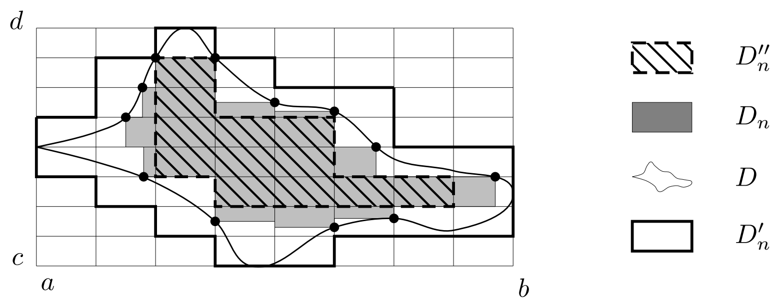
Proof. Because D is bounded there is a rectangle $[a,b]\times[c,d]$ which includes D. We can choose for example

$$a = \min_{(u,v)\in D} u, \qquad b = \max_{(u,v)\in D} u, \qquad c = \min_{(u,v)\in D} v \quad \text{and} \quad d = \max_{(u,v)\in D} v.$$

Divide the rectangle $[a,b]\times[c,d]$ into n^2 rectangles of sides $\frac{b-a}{n}$ and $\frac{d-c}{n}$. Consider D'_n the set of points of rectangles which have common points with D and D''_n the set of points of rectangles which are entirely contained in D. We have $D''_n \subseteq D \subseteq D'_n$. Because C is simple and piecewise smooth the set D''_n contains for sufficiently large n at least two rectangles.

Also, because C is piecewise smooth it has finite length L. Consider $\delta = \min\left(\frac{b-a}{n}, \frac{d-c}{n}\right)$. Because every arc of C of length at most δ is contained in at most four rectangles from $D'_n \setminus D''_n$, we deduce that $D'_n \setminus D''_n$ has at most $4\left(\left[\frac{L}{\delta}\right]+1\right)$ rectangles. This is a well-known result [5].

We want to find a domain $D_n \subseteq D'_n$ satisfying the property from Remark 2.2 and a curve C_n, the boundary of D_n, such that this curve is simple and intersects the curve C in such a way that the distance between consecutive points of intersection tends to zero.



This can be accomplished in many ways. For example, consider C''_n the boundary of D''_n. Let $R_1, R_2, \ldots, R_N$ be the rectangles from D''_n which have in common with C''_n at least one side. Let $i \in \{1,\ldots,N\}$. If R_i has common points with C choose M_i one point of the intersection. Consider now, one rectangle R_i which has only one exterior side (only one side in common with C''_n). We lengthen the two sides of the rectangle which have common vertices with the exterior side, keeping fixed the third interior side. In this way we enlarge the rectangle until it touches the curve C in at least one point M_i. If the rectangle R_i has two or three exterior sides we choose one of these sides and enlarge the rectangle as above. In this manner, we obtain an "enlarged" domain D_n, with the boundary C_n and the points $M_1, M_2, \ldots, M_N$ with the property that two consecutive points are at a distance at most $4\cdot\max\left(\frac{b-a}{n}, \frac{d-c}{n}\right)$ one from another.

Using Remark 2.2 we have

$$\int_{C_n} P\cdot\frac{\partial x}{\partial u}\,\mathrm{d}u + P\cdot\frac{\partial x}{\partial v}\,\mathrm{d}v = \iint_{D_n}\left(\frac{\partial P}{\partial u}\cdot\frac{\partial x}{\partial v} - \frac{\partial P}{\partial v}\cdot\frac{\partial x}{\partial u}\right)\mathrm{d}u\,\mathrm{d}v. \tag{2.2}$$

Let $\varepsilon > 0$. Because

$$Area(D\setminus D_n) \le Area(D'_n\setminus D''_n) \le \underbrace{4\left(\left[\frac{L}{\delta}\right]+1\right)\cdot\frac{(b-a)(d-c)}{n^2}}_{\text{tends to 0 when } n \text{ tends to infinity}}$$

and because of the integrability of $\frac{\partial P}{\partial u} \cdot \frac{\partial x}{\partial v} - \frac{\partial P}{\partial v} \cdot \frac{\partial x}{\partial u}$ we have

$$\left| \iint_D \left(\frac{\partial P}{\partial u} \cdot \frac{\partial x}{\partial v} - \frac{\partial P}{\partial v} \cdot \frac{\partial x}{\partial u} \right) \mathrm{d}u \, \mathrm{d}v - \iint_{D_n} \left(\frac{\partial P}{\partial u} \cdot \frac{\partial x}{\partial v} - \frac{\partial P}{\partial v} \cdot \frac{\partial x}{\partial u} \right) \mathrm{d}u \, \mathrm{d}v \right| \leq \varepsilon.$$

For the points $M_i(u_i, v_i)$, $i = 1, \ldots, N+1$, with $M_{N+1} = M_1$ using the definition of the line integral along C for sufficiently large n we get

$$\left| \int_C P \cdot \frac{\partial x}{\partial u} \mathrm{d}u + P \cdot \frac{\partial x}{\partial v} \mathrm{d}v - \sum_{k=1}^{N+1} P(M_i) \cdot \frac{\partial x}{\partial u}(M_i)(u_{i+1} - u_i) + P(M_i) \cdot \frac{\partial x}{\partial v}(M_i)(v_{i+1} - v_i) \right| \leq \varepsilon.$$

The same we get for the curve C_n

$$\left| \int_{C_n} P \cdot \frac{\partial x}{\partial u} \mathrm{d}u + P \cdot \frac{\partial x}{\partial v} \mathrm{d}v - \sum_{k=1}^{N+1} P(M_i) \cdot \frac{\partial x}{\partial u}(M_i)(u_{i+1} - u_i) + P(M_i) \cdot \frac{\partial x}{\partial v}(M_i)(v_{i+1} - v_i) \right| \leq \varepsilon.$$

So, we deduce that

$$\left| \int_C P \cdot \frac{\partial x}{\partial u} \mathrm{d}u + P \cdot \frac{\partial x}{\partial v} \mathrm{d}v - \iint_D \left(\frac{\partial P}{\partial u} \cdot \frac{\partial x}{\partial v} - \frac{\partial P}{\partial v} \cdot \frac{\partial x}{\partial u} \right) \mathrm{d}u \, \mathrm{d}v. \right| \leq 3\varepsilon.$$

Letting $\varepsilon \searrow 0$ we obtain relation (2.1). □

Theorem 2.1. *Let $\vec{r} = x(u, v)\vec{\imath} + y(u, v)\vec{\jmath} + z(u, v)\vec{k}$, $(u, v) \in D$ be the parametrization of a simple, open and orientable surface S, with $x, y, z \in C^1(D)$, where D is a closed bounded simply connected domain in the plane with its boundary ∂D, a simple closed positively oriented piecewise smooth curve, and $V = P(x, y, z)\vec{\imath} + Q(x, y, z)\vec{\jmath} + R(x, y, z)\vec{k}$ be a vector field such that P, Q and R are bounded functions which have only finitely many points of discontinuity in S and which have partial derivatives which are integrable on S. Then*

$$\int_C P \, \mathrm{d}x + Q \, \mathrm{d}y + R \, \mathrm{d}z = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathrm{d}y \, \mathrm{d}z + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathrm{d}z \, \mathrm{d}x + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y,$$

where C is the boundary curve of S and the image of ∂D through the given parametrization.

Proof. We prove that

$$\int_C P \, \mathrm{d}x = \iint_S \frac{\partial P}{\partial z} \mathrm{d}z \, \mathrm{d}x - \frac{\partial P}{\partial y} \mathrm{d}x \, \mathrm{d}y.$$

Similarly

$$\int_C Q \, \mathrm{d}y = \iint_S \frac{\partial Q}{\partial x} \mathrm{d}x \, \mathrm{d}y - \frac{\partial Q}{\partial z} \mathrm{d}y \, \mathrm{d}z,$$
$$\int_C R \, \mathrm{d}z = \iint_S \frac{\partial R}{\partial y} \mathrm{d}y \, \mathrm{d}z - \frac{\partial R}{\partial x} \mathrm{d}z \, \mathrm{d}x.$$

Adding the three relations we obtain Stokes formula.

Because $\mathrm{d}x = \frac{\partial x}{\partial u}\,\mathrm{d}u + \frac{\partial x}{\partial v}\,\mathrm{d}v$ we obtain

$$\int_C P\,\mathrm{d}x = \int_{\partial D} P\left(\frac{\partial x}{\partial u}\,\mathrm{d}u + \frac{\partial x}{\partial v}\,\mathrm{d}v\right) = \int_{\partial D} P\cdot\frac{\partial x}{\partial u}\,\mathrm{d}u + P\cdot\frac{\partial x}{\partial v}\,\mathrm{d}v.$$

Applying Lemma 2.2 we have

$$\int_{\partial D} P\cdot\frac{\partial x}{\partial u}\,\mathrm{d}u + P\cdot\frac{\partial x}{\partial v}\,\mathrm{d}v = \iint_D \left(\frac{\partial P}{\partial u}\cdot\frac{\partial x}{\partial v} - \frac{\partial P}{\partial v}\cdot\frac{\partial x}{\partial u}\right)\mathrm{d}u\,\mathrm{d}v.$$

Because $P = P(x(u,v), y(u,v), z(u,v))$ we have

$$\begin{aligned}
\frac{\partial P}{\partial u} &= \frac{\partial P}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial P}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial P}{\partial z}\cdot\frac{\partial z}{\partial u},\\
\frac{\partial P}{\partial v} &= \frac{\partial P}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial P}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial P}{\partial z}\cdot\frac{\partial z}{\partial v},
\end{aligned}$$

so

$$\begin{aligned}
&\iint_D \left(\frac{\partial P}{\partial u}\cdot\frac{\partial x}{\partial v} - \frac{\partial P}{\partial v}\cdot\frac{\partial x}{\partial u}\right)\mathrm{d}u\,\mathrm{d}v\\
&\qquad= \iint_D \left(\frac{\partial P}{\partial u} - \frac{\partial P}{\partial x}\frac{\partial x}{\partial u}\right)\cdot\frac{\partial x}{\partial v} - \left(\frac{\partial P}{\partial v} - \frac{\partial P}{\partial x}\frac{\partial x}{\partial v}\right)\cdot\frac{\partial x}{\partial u}\,\mathrm{d}u\,\mathrm{d}v\\
&\qquad= \iint_D \left(\frac{\partial P}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial u}\right)\cdot\frac{\partial x}{\partial v} - \left(\frac{\partial P}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial v}\right)\cdot\frac{\partial x}{\partial u}\,\mathrm{d}u\,\mathrm{d}v\\
&\qquad= \iint_D \frac{\partial P}{\partial z}\left(\frac{\partial z}{\partial u}\cdot\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\cdot\frac{\partial x}{\partial u}\right) - \frac{\partial P}{\partial y}\left(\frac{\partial y}{\partial v}\cdot\frac{\partial x}{\partial u} - \frac{\partial y}{\partial u}\cdot\frac{\partial x}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v\\
&\qquad= \iint_D \frac{\partial P}{\partial z}\cdot\begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}\\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\end{vmatrix} - \frac{\partial P}{\partial y}\cdot\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}\,\mathrm{d}u\,\mathrm{d}v\\
&\qquad= \iint_S \frac{\partial P}{\partial z}\,\mathrm{d}z\,\mathrm{d}x - \frac{\partial P}{\partial y}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

□

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