Uniform approximation of functions by Bernstein-type operators

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Abstract

For the class of bounded functions defined on $[0, 1]$ and continuous on $(0, 1)$ we give a characterization of the functions which can be uniformly approximated by Bernstein-type operators.

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1. Introduction

Bernstein polynomials were introduced by S. N. Bernstein [1] in 1912 to constructively solve the problem of K. Weierstrass [13] of uniformly approximating the continuous functions by using polynomials. They are defined by

$$B_n(f, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0, 1], \quad n \geq 1 \quad (1)$$

They approximate uniformly every continuous function $f$ defined on the compact $[0, 1]$, i.e.

$$\|B_n f - f\| = \sup_{x \in [0,1]} |B_n(f, x) - f(x)| \to 0, \quad \text{when } n \to \infty.$$

We study in this paper a general class of Bernstein type operators:

$$L_n(f, x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \Lambda_{n,k}(f), \quad n \geq 1 \quad (2)$$
where \( \Lambda_{n,k}(f) \) are positive linear functionals such that

\[
\Lambda_{n,k}(1) = 1, \quad \text{for every } k = 0, 1, \ldots, n.
\]

\[
\lim_{n \to \infty} \|L_n e_1 - e_1\| = 0,
\]

\[
\lim_{n \to \infty} \|L_n e_2 - e_2\| = 0.
\]

Using the Theorem of Popoviciu-Bohman-Korovkin [10, 2, 7] one can prove that every continuous function \( f \in C[0,1] \) can be uniformly approximated by \( L_n f \).

The problem studied in this paper is the following: if we restrict to the class of bounded functions defined on \([0,1]\) and continuous on \((0,1)\) does the uniform approximation property of \( L_n \) operators still hold? It is possible to uniformly approximate \( \sin \frac{1}{x} \)? We give in Theorem 1 the characterization of the functions from this class which can be uniformly approximated by \( L_n \).

2. Main result

**Theorem 1.** If \( f : [0,1] \to \mathbb{R} \) is a bounded function which is continuous on \((0,1)\) then

\[ \|L_n f - f\| \to 0, \quad \text{when } n \to \infty \]

if and only if

\[ f \text{ is uniformly continuous on } (0,1). \]

**Proof.** Suppose \( \|L_n f - f\| \to 0, \) when \( n \to \infty. \) We prove that \( f \) must be uniformly continuous on \((0,1).\)

Let us denote by \( p_{n,k} \) the polynomials \( \binom{n}{k} x^k (1-x)^{n-k} \) . We have (see [4, p. 305])

\[ p_{n,k}'(x) = n [p_{n-1,k-1}(x) - p_{n-1,k}(x)]. \]

So,

\[ L'_n(f, x) = n \sum_{k=1}^{n} p_{n-1,k-1}(x) \Lambda_{n,k}(f) - n \sum_{k=0}^{n-1} p_{n-1,k}(x) \Lambda_{n,k}(f). \]
Because $\sum_{k=0}^{n} p_{n,k}(x) = 1$, we deduce that $|L'_n(f, x)| \leq 2n \| f \|$. Using the properties of the global modulus of continuity (see [4]) we have

$$\omega(f, \delta_n) \leq \omega(f - L_n f, \delta_n) + \omega(L_n f, \delta_n)$$

$$\leq 2\|f - L_n f\| + \sup_{|t-x| \leq \delta_n} |L_n(f, t) - L_n(f, x)|$$

$$\leq 2\|f - L_n f\| + \delta_n \sup_{c \in (0,1)} |L'_n(f, c)| \leq 2\|f - L_n f\| + 2\|f\| n \delta_n.$$

If we choose the sequence $\delta_n$ such that $\delta_n \cdot n$ tends to zero, we deduce from the above inequality that $\omega(f, \delta_n) \to 0$ when $n \to \infty$. This proves that $f$ is uniformly continuous on $(0,1)$.

The converse can be obtained using the Shisha-Mond [11] evaluation of the rate of approximation:

$$|L_n(f, x) - f(x)| \leq 2 \cdot \omega(f, \sqrt{L_n(|t-x|^2, x)}).$$

Let us denote

$$\delta_n = \sup_{x \in (0,1)} \sqrt{L_n(|t-x|^2, x)}$$

Using the relations (3) and because

$$\delta_n \leq \sqrt{\|L_n e_2 - e_2\| + 2\|L_n e_1 - e_1\|}$$

we deduce that $\delta_n$ tends to zero. Because $f$ is uniformly continuous we obtain that $\omega(f, \delta_n)$ tends to zero, so $\|L_n f - f\| \to 0$ tends to zero, when $n$ tends to infinity. \hfill \Box

**Example 2.** The function $f(x) = \sin \frac{1}{x}$ for $x \in (0,1)$ cannot be uniformly approximated by the Bernstein-type operators in the uniform norm.

**Example 3.** The result of Theorem 1 is true for Bernstein-Stancu operators and in particular for Bernstein operators. Indeed, for $\Lambda_{n,k}(f) = f \left( \frac{k+\alpha}{n+\beta} \right)$, where $0 \leq \alpha \leq \beta$ we obtain the operators introduced and studied by D. D. Stancu [12].

**Example 4.** For

$$\Lambda_{n,k}(f) = (n + 1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) \, dt$$

we obtain the operators of Kantorovich [6] for which the Theorem 1 is true.
Example 5. For

\[ \Lambda_{n,k}(f) = (n + 1) \int_0^1 \binom{n}{k} t^k (1 - t)^{n-k} f(t) \, dt \]

we obtain the operators introduced by J. L. Durrmeyer [5] in 1967 and studied by A. Lupas [8] and M. M. Derriennic [3]. Theorem 1 is true for these operators, too.

References


