Uniform approximation of functions by Bernstein-type operators

Adrian Holhoş

Department of Mathematics, Technical University of Cluj-Napoca, Str. Memorandumului nr. 28, 400114 Cluj-Napoca, Romania

---

Abstract

For the class of bounded functions defined on $[0,1]$ and continuous on $(0,1)$ we give a characterization of the functions which can be uniformly approximated by Bernstein-type operators.

Keywords: positive linear operator, modulus of continuity, weighted spaces
2000 MSC: 41A36, 41A25

---

1. Introduction

Bernstein polynomials were introduced by S. N. Bernstein [1] in 1912 to constructively solve the problem of K. Weierstrass [13] of uniformly approximating the continuous functions by using polynomials. They are defined by

$$B_n(f,x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right),\quad x\in[0,1],\ n\geq 1 \tag{1}$$

They approximate uniformly every continuous function f defined on the compact $[0,1]$, i.e.

$$\|B_nf-f\|=\sup_{x\in[0,1]}|B_n(f,x)-f(x)|\to 0,\qquad \text{when } n\to\infty.$$

We study in this paper a general class of Bernstein type operators:

$$L_n(f,x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\Lambda_{n,k}(f),\quad n\geq 1, \tag{2}$$

where $\Lambda_{n,k}(f)$ are positive linear functionals such that

$$\Lambda_{n,k}(1) = 1, \quad \text{for every } k = 0, 1, \ldots, n.$$
$$\lim_{n\to\infty} \|L_n e_1 - e_1\| = 0, \tag{3}$$
$$\lim_{n\to\infty} \|L_n e_2 - e_2\| = 0.$$

Using the Theorem of Popoviciu-Bohman-Korovkin [10, 2, 7] one can prove that every continuous function $f \in C[0,1]$ can be uniformly approximated by $L_n f$.

The problem studied in this paper is the following: if we restrict to the class of bounded functions defined on $[0,1]$ and continuous on $(0,1)$ does the uniform approximation property of L_n operators still hold? It is possible to uniformly approximate $\sin\frac{1}{x}$? We give in Theorem 1 the characterization of the functions from this class which can be uniformly approximated by L_n.

2. Main result

Theorem 1. *If* $f\colon [0,1] \to \mathbb{R}$ *is a bounded function which is continuous on* $(0,1)$ *then*

$$\|L_n f - f\| \to 0, \text{ when } n \to \infty$$

if and only if

$$f \text{ is uniformly continuous on } (0,1).$$

Proof. Suppose $\|L_n f - f\| \to 0$, when $n \to \infty$. We prove that f must be uniformly continuous on $(0,1)$.

Let us denote by $p_{n,k}$ the polynomials $\binom{n}{k} x^k (1-x)^{n-k}$ .We have (see [4, p. 305])

$$p'_{n,k}(x) = n\left[p_{n-1,k-1}(x) - p_{n-1,k}(x)\right].$$

So,

$$L'_n(f,x) = n\sum_{k=1}^{n} p_{n-1,k-1}(x)\Lambda_{n,k}(f) - n\sum_{k=0}^{n-1} p_{n-1,k}(x)\Lambda_{n,k}(f).$$

Because $\sum_{k=0}^{n} p_{n,k}(x) = 1$, we deduce that $|L'_n(f,x)| \leq 2n \|f\|$. Using the properties of the global modulus of continuity (see [4]) we have

$$\begin{aligned}\omega(f,\delta_n) &\leq \omega(f - L_n f, \delta_n) + \omega(L_n f, \delta_n)\\ &\leq 2\|f - L_n f\| + \sup_{\substack{|t-x|\leq\delta_n\\ t,x\in(0,1)}} |L_n(f,t) - L_n(f,x)|\\ &\leq 2\|f - L_n f\| + \delta_n \sup_{c\in(0,1)} |L'_n(f,c)| \leq 2\|f - L_n f\| + 2\|f\|\, n\, \delta_n.\end{aligned}$$

If we choose the sequence δ_n such that $\delta_n \cdot n$ tends to zero, we deduce from the above inequality that $\omega(f,\delta_n) \to 0$ when $n \to \infty$. This proves that f is uniformly continuous on $(0,1)$.

The converse can be obtained using the Shisha-Mond [11] evaluation of the rate of approximation:

$$|L_n(f,x) - f(x)| \leq 2 \cdot \omega\left(f, \sqrt{L_n(|t-x|^2, x)}\right).$$

Let us denote

$$\delta_n = \sup_{x\in(0,1)} \sqrt{L_n(|t-x|^2, x)}$$

Using the relations (3) and because

$$\delta_n \leq \sqrt{\|L_n e_2 - e_2\| + 2\|L_n e_1 - e_1\|}$$

we deduce that δ_n tends to zero. Because f is uniformly continuous we obtain that $\omega(f,\delta_n)$ tends to zero, so $\|L_n f - f\| \to 0$ tends to zero, when n tends to infinity. □

Example 2. The function $f(x) = \sin \frac{1}{x}$ for $x \in (0,1)$ cannot be uniformly approximated by the Bernstein-type operators in the uniform norm.

Example 3. The result of Theorem 1 is true for Bernstein-Stancu operators and in particular for Bernstein operators. Indeed, for $\Lambda_{n,k}(f) = f\left(\frac{k+\alpha}{n+\beta}\right)$, where $0 \leq \alpha \leq \beta$ we obtain the operators introduced and studied by D. D. Stancu [12].

Example 4. For

$$\Lambda_{n,k}(f) = (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\, \mathrm{d}t$$

we obtain the operators of Kantorovich [6] for which the Theorem 1 is true.

Example 5. For

$$\Lambda_{n,k}(f) = (n+1)\int_0^1 \binom{n}{k} t^k(1-t)^{n-k} f(t)\,\mathrm{d}t$$

we obtain the operators introduced by J. L. Durrmeyer [5] in 1967 and studied by A. Lupaş [8] and M. M. Derriennic [3]. Theorem 1 is true for these operators, too.

References

[1] S. N. Bernstein, *Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités*, Commun. Soc. Math. Kharkow (2), **13** (1912-1913), 1–2.

[2] H. Bohman, *On approximation of continuous and of analytic functions*, Ark. Mat., **2** (1952), 43–56.

[3] M. M. Derriennic, *Sur l'"approximation des fonctions intégrables sur* $[0,1]$ *par des polynomes de Bernstein modifiés*, J. Approx. Theory, **31** (1981), 325–343.

[4] R. A. DeVore and G. G. Lorentz, *Constructive Approximation*, Springer-Verlag, Berlin-Heidelberg-New York, 1993.

[5] J. L. Durrmeyer, *Une formule d'"inversion de la transformée de Laplace: Application à la théorie des moments*, Thése de 3e cycle, Faculté des Sciences de l'"Université de Paris, 1967.

[6] L.V. Kantorovich, *Sur certains développements suivant les polynomes de la forme de S. Bernstein*, I, II, C.R. Acad. URSS (1930), 563–568, 595–600.

[7] P. P. Korovkin, *On the convergence of linear positive operators in the space of continuous functions*, Dokl. Akad. Nauk, **90** (1953), 961–964 (in Russian).

[8] A. Lupaş, *Die Folge der Betaoperatoren*, Dissertation, Universität Stuttgart, 1972.

[9] P. E. Parvanov, *Weighted Approximation by a Class of Bernstein-Type Operators*, Math. Balkanica (N.S.), **25** (2011), fasc. 1-2, 31–38.

[10] T. Popoviciu, *Asupra demonstraţiei teoremei lui Weierstrass cu ajutorul polinoamelor de interpolare*, în Lucrările Sesiunii Generale Ştiinţifice din 2-12 iunie 1950, 1664–1667, Editura Academiei Republicii Populare Române, Bucureşti, 1951. (in Romanian)

[11] O. Shisha and B. Mond, *The Degree of Convergence of Sequences of Linear Positive Operators*, Proc. Natl. Acad. Sci. USA, **60** (1968), 1196–1200.

[12] D. D. Stancu, *Asupra unei generalizări a polinoamelor lui Bernstein*, Stud. Univ. Babeş-Bolyai Math., **14** (1969), 31–45. (in Romanian)

[13] K. Weierstrass, *Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen*, Sitzungsber. Akad. Berlin (1885), 633–639, 789–805.