Seminar 5

Serii Taylor

Probleme rezolvate

Problema 5.1. Să se dezvolte în serie Taylor următoarele funcţii în jurul punctelor indicate, stabilind şi mulţimea de valabilitate pentru dezvoltările obţinute

$$a)\quad f(x)=\frac{1}{2x+3},\quad x_0=1 \qquad\qquad b)\quad f(x)=\frac{3x}{x^2+5x+6},\quad x_0=0$$

$$c)\quad f(x)=\frac{x^2}{(1+x^2)^5},\quad x_0=0 \qquad\qquad d)\quad f(x)=\frac{1}{\sqrt{4-x^2}},\quad x_0=0$$

$$e)\quad f(x)=\ln\frac{2-x}{3+x},\quad x_0=0 \qquad\qquad f)\quad f(x)=\operatorname{sh}^2\frac{x}{2},\quad x_0=0$$

$$g)\quad f(x)=\sin 2x\cdot\cos 3x,\quad x_0=0 \qquad\qquad h)\quad f(x)=\operatorname{arctg} x,\quad x_0=0$$

$$i)\quad f(x)=x\ln(x+\sqrt{x^2+9}),\quad x_0=0 \qquad\qquad j)\quad f(x)=\int_0^x\frac{\sin t}{t}\,\mathrm{d}t,\quad x_0=0$$

$$k)\quad f(x)=(x^2-x+1)e^{-x},\quad x_0=0$$

Pentru funcţia de la punctul k) să se calculeze $f^{(2021)}(0)$.

Soluţie 5.1. a) Facem schimbarea de variabilă $u=x-x_0=x-1$ pentru a obţine o dezvoltare în jurul lui 0. Folosind seria geometrică avem

$$\begin{aligned}\frac{1}{2x+3}&=\frac{1}{2(u+1)+3}=\frac{1}{2u+5}=\frac{1}{5}\cdot\frac{1}{1+\frac{2u}{5}}=\frac{1}{5}\cdot\frac{1}{1-\left(-\frac{2u}{5}\right)}\\&=\frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{2u}{5}\right)^n=\frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{2}{5}\right)^n u^n=\sum_{n=0}^{\infty}\frac{(-2)^n}{5^{n+1}}(x-1)^n.\end{aligned}$$

Din condiţia $\left|-\frac{2u}{5}\right|<1$, rezultă $|u|<\frac{5}{2}$, care este echivalent cu $x\in\left(-\frac{3}{2},\frac{7}{2}\right)$, care este mulţimea de valabilitate a seriei.

b) Despărţim în fracţii simple

$$f(x)=\frac{3x}{x^2+5x+6}=\frac{3x}{(x+2)(x+3)}=\frac{A}{x+2}+\frac{B}{x+3},$$

cu $A=-6$ şi $B=9$. Folosim din nou seria geometrică

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\cdots,\quad |x|<1$$

şi obţinem

$$f(x) = \frac{-6}{x+2} + \frac{9}{x+3} = \frac{-3}{1+\frac{x}{2}} + \frac{3}{1+\frac{x}{3}} = -3\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n + 3\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n$$
$$= \sum_{n=0}^{\infty}\left[\frac{3(-1)^{n+1}}{2^n} + \frac{(-1)^n}{3^{n-1}}\right]x^n$$

Din condiţiile $\left|-\frac{x}{2}\right|$ şi $\left|-\frac{x}{3}\right|$ obţinem $x \in (-2,2)$, mulţimea de valabilitate a seriei.

c) Folosim seria binomială

$$(1+x)^\alpha = \sum_{n=0}^{\infty}\binom{\alpha}{n}x^n, \quad |x|<1, \qquad \text{unde } \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

Avem

$$f(x) = \frac{1+x^2-1}{(1+x^2)^5} = \frac{1}{(1+x^2)^4} - \frac{1}{(1+x^2)^5} = (1+x^2)^{-4} - (1+x^2)^{-5}$$
$$= \sum_{n=0}^{\infty}\binom{-4}{n}x^{2n} - \sum_{n=0}^{\infty}\binom{-5}{n}x^{2n} = \sum_{n=0}^{\infty}\left[\binom{-4}{n} - \binom{-5}{n}\right]x^{2n}$$

Pentru că

$$\binom{-4}{n} = \frac{(-4)(-5)\cdots(-4-n+1)}{n!} = \frac{(-1)^n 4\cdot 5\cdots(n+3)}{n!} = \frac{(-1)^n(n+1)(n+2)(n+3)}{6}$$

şi

$$\binom{-5}{n} = \frac{(-5)(-6)\cdots(-5-n+1)}{n!} = \frac{(-1)^n(n+1)(n+2)(n+3)(n+4)}{24}$$

obţinem dezvoltarea

$$f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)(n+3)}{24}\left(4-(n+4)\right)x^{2n}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}n(n+1)(n+2)(n+3)}{24}x^{2n}.$$

Din condiţia $|x^2| = |x|^2 < 1$ rezultă $|x|<1$, adică $x \in (-1,1)$.

d) Folosim seria binomială

$$f(x) = (4-x^2)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}} = \frac{1}{2}\left(1+\left(-\frac{x^2}{4}\right)\right)^{-\frac{1}{2}}$$
$$= \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x^2}{4}\right)^n = \sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}\left(-\frac{x^2}{4}\right)^n$$
$$= 1 + \sum_{n=1}^{\infty}\frac{1\cdot 3\cdots(2n-1)}{8^n n!}x^{2n}.$$

Dezvoltarea este valabilă pentru $\left|-\frac{x^2}{4}\right| < 1$, adică $x \in (-2,2)$.

e) Folosim dezvoltarea în serie a funcţiei logaritm

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots, \quad |x|<1$$

şi obţinem

$$\begin{aligned} f(x) &= \ln(2-x) - \ln(3+x) = \ln 2 + \ln\left(1 - \frac{x}{2}\right) - \ln 3 - \ln\left(1 + \frac{x}{3}\right) \\ &= \ln 2 - \sum_{n=1}^{\infty} \frac{x^n}{2^n n} - \ln 3 - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{3^n n} = \ln\frac{2}{3} - \sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^{n-1}}{3^n}\right) \frac{x^n}{n}. \end{aligned}$$

Din condiţiile $\left|-\frac{x}{2}\right|$ şi $\left|\frac{x}{3}\right|$ rezultă $x \in (-2, 2)$.

f) Funcţia sinus hiperbolic este definită prin

$$\operatorname{sh} x = \frac{e^x - e^{-x}}{2}.$$

Folosind dezvoltarea în serie a funcţiei exponenţiale

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots,$$

va rezulta

$$\begin{aligned} f(x) &= -\frac{1}{2} + \frac{1}{4} e^x + \frac{1}{4} e^{-x} = -\frac{1}{2} + \frac{1}{4} \sum_{n=0}^{\infty} \frac{x^n}{n!} + \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \\ &= \frac{1}{4} \sum_{n=1}^{\infty} \frac{x^n}{n!} \left[1 + (-1)^n\right] = \frac{1}{2} \sum_{p=1}^{\infty} \frac{x^{2p}}{(2p)!}. \end{aligned}$$

g) Folosind dezvoltarea în serie a funcţiei sinus:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{5!} - \cdots,$$

obţinem

$$f(x) = \frac{1}{2} \left(\sin 5x - \sin x\right) = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{5^{2n+1} - 1}{(2n+1)!} x^{2n+1}.$$

h) Derivata funcţiei se poate dezvolta în serie în felul următor:

$$f'(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Va rezulta prin integrare că

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C.$$

Pentru $x = 0$ avem $f(0) = C$. Dar $f(0) = \operatorname{arctg} 0 = 0$. Obţinem $C = 0$ şi

$$\operatorname{arctg} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

□

i) Pornim de la

$$[\ln(x+\sqrt{x^2+9})]' = \frac{1}{\sqrt{x^2+9}} = (9+x^2)^{-\frac{1}{2}} = 9^{-\frac{1}{2}}\left(1+\frac{x^2}{9}\right)^{-\frac{1}{2}} = \frac{1}{3}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x^2}{9}\right)^n,$$

dezvoltare valabilă pentru $x \in (-3,3)$, obţinem prin integrare

$$\ln(x+\sqrt{x^2+9}) = \frac{1}{3}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{1}{9}\right)^n \cdot \frac{x^{2n+1}}{2n+1} + C.$$

Pentru $x=0$ se obţine $\ln 3 = 0 + C$, adică $C = \ln 3$. Ţinând cont că

$$\binom{-\frac{1}{2}}{n} = \frac{\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!} = \frac{1\cdot 3\cdots(2n-1)}{2^n n!} = \frac{(2n)!}{4^n (n!)^2}$$

obţinem

$$f(x) = x\ln(x+\sqrt{x^2+9}) = \frac{1}{3}\sum_{n=0}^{\infty}\frac{(2n)!}{36^n(n!)^2}\cdot\frac{x^{2n+1}}{2n+2} + \ln 3, \quad x \in (-3,3).$$

j) Folosind dezvoltarea lui $\sin t$ obţinem

$$f(x) = \int_0^x \frac{1}{t}\cdot \sin t \,\mathrm{d}t = \int_0^x \frac{1}{t}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}t^{2n+1}\,\mathrm{d}t = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\int_0^x t^{2n}\,\mathrm{d}t$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!(2n+1)}\cdot x^{2n+1}, \quad x \in \mathbb{R}.$$

k) Avem

$$f(x) = (x^2-x+1)e^{-x} = (x^2-x+1)\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{n!} - \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n!} + \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$$
$$= \sum_{m=2}^{\infty}\frac{(-1)^{m-2} x^m}{(m-2)!} - \sum_{p=1}^{\infty}\frac{(-1)^{p-1} x^p}{(p-1)!} + \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$$
$$= \sum_{k=2}^{\infty}\frac{(-1)^{k-2} x^k}{(k-2)!} - x - \sum_{k=2}^{\infty}\frac{(-1)^{k-1} x^k}{(k-1)!} + 1 - x + \sum_{k=2}^{\infty}\frac{(-1)^k x^k}{k!}$$
$$= 1 - 2x + \sum_{k=2}^{\infty}\frac{(-1)^k x^k}{k!}\left(k^2+1\right), \quad x \in \mathbb{R}.$$

Coeficientul lui x^k din seria Taylor este $\frac{f^{(k)}(0)}{k!}$. Se obţine

$$f^{(2021)}(0) = 2021!\frac{(-1)^{2021}(2021^2+1)}{2021!} = -(2021^2+1).$$

Probleme propuse

5.2. Să se dezvolte în serie Taylor următoarele funcţii în jurul punctelor indicate, precizând şi mulţimea de valabilitate pentru dezvoltarea obţinută

a) $f(x) = \dfrac{1}{3x-2}, \quad x_0 = -1$

b) $f(x) = \sqrt{1+x}, \quad x_0 = 1$

c) $f(x) = \dfrac{1}{(x-1)(3x+1)}, \quad x_0 = 0$

d) $f(x) = \dfrac{1}{2}\ln\dfrac{1-x}{1+x}, \quad x_0 = 0$

e) $f(x) = \cos^2 x, \quad x_0 = 0$

f) $f(x) = \ln(x + \sqrt{1+x^2}), \quad x_0 = 0$

g) $f(x) = e^{-x^2}, \quad x_0 = 0$

h) $f(x) = x \arcsin x, \quad x_0 = 0.$

Indicaţii la problemele propuse

5.2. a)

$$\frac{1}{3x-2} = -\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n (x+1)^n, \quad x \in \left(-\frac{8}{3}, \frac{2}{3}\right).$$

$$b)\ \sqrt{1+x} = \sqrt{2}\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}1\cdot 3\cdots(2n-3)}{n!\ 4^n}(x-1)^n\right), \quad x \in (-1,3).$$

$$c)\ \frac{1}{(x-1)(3x+1)} = \frac{1}{4(x-1)} - \frac{3}{4(3x+1)} = -\frac{1}{4}\sum_{n=0}^{\infty}\left(1-(-3)^{n+1}\right)x^n, \quad x \in \left(-\frac{1}{3}, \frac{1}{3}\right).$$

$$d)\ \frac{1}{2}\ln\frac{1-x}{1+x} = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}, \quad x \in (-1,1).$$

$$e)\ \cos^2 x = \frac{1+\cos 2x}{2} = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n-1}}{(2n)!}x^{2n}, \quad x \in \mathbb{R}.$$

f) Avem

$$f'(x) = \frac{1}{\sqrt{1+x^2}} = \sum_{n=0}^{\infty} \mathrm{C}^n_{-\frac{1}{2}} x^{2n},$$

de unde

$$f(x) = x + \sum_{n=1}^{\infty}\frac{(-1)^n 1\cdot 3\cdots(2n-1)}{2^n n!}\,\frac{x^{2n+1}}{2n+1}, \quad x \in (-1,1).$$

$$g)\ e^{-x^2} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}, \quad x \in \mathbb{R}.$$

$$h)\ x\arcsin x = \sum_{n=0}^{\infty}\frac{1\cdot 3\cdots(2n-1)}{2^n n!}\,\frac{x^{2n+2}}{2n+1}, \quad x \in (-1,1).$$