Seminar 12

Integrale improprii

Să se studieze convergenţa integralelor şi în caz de convergenţă să se determine valoarea acestora

12.1. $\displaystyle\int_0^\infty \frac{x}{(x+1)(x+2)}\,\mathrm{d}x$

12.2. $\displaystyle\int_0^\infty \frac{x^{\sqrt{2}}\,\mathrm{d}x}{x^2+1}.$

12.3. $\displaystyle\int_0^\infty \frac{x}{(x+1)(x+2)^2}\,\mathrm{d}x$

12.4. $\displaystyle\int_0^1 \frac{x(\arcsin x)^2}{\sqrt{1-x^2}}\,\mathrm{d}x.$

12.5. $\displaystyle\int_0^{\frac{\pi}{2}} \ln(\sin x)\,\mathrm{d}x.$

12.6. $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{1-x^2}}\,\mathrm{d}x.$

12.7. $\displaystyle\int_0^\infty \frac{\ln x}{x^2+1}\,\mathrm{d}x.$

12.8. $\displaystyle\int_0^\infty \frac{\operatorname{arctg} x}{x^2+x+1}\,\mathrm{d}x.$

Indicaţii la problemele propuse

12.1. $\alpha = 1$ DIV

12.2. $\alpha = 2-\sqrt{2} < 1$ DIV

12.3. $\alpha = 2 > 1$ CONV. Pentru calcul se descompune în fracţii simple

$$\frac{x}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

cu $A=-1$, $B=1$, $C=2$. Avem

$$\begin{aligned}\int_0^\infty \frac{x}{(x+1)(x+2)^2}\,\mathrm{d}x &= \lim_{t\to\infty}\int_0^t \frac{x}{(x+1)(x+2)^2}\,\mathrm{d}x\\ &= \lim_{t\to\infty}\int_0^t \frac{-1}{x+1}\,\mathrm{d}x + \int_0^t \frac{1}{x+2}\,\mathrm{d}x + \int_0^t \frac{2}{(x+2)^2}\,\mathrm{d}x\\ &= \lim_{t\to\infty} \ln\frac{t+2}{t+1} - \ln 2 - \frac{2}{t+2} + 1 = 1 - \ln 2.\end{aligned}$$

12.4. $\alpha = \frac{1}{2}$ CONV. Pentru calcul facem substutuţia $x = \sin t$. Se obţine

$$\int_0^1 \frac{x(\arcsin x)^2}{\sqrt{1-x^2}}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} t^2 \sin t\,\mathrm{d}t = \pi - 2.$$

12.5. $\alpha = 1/2$ CONV Pentru a o calcula facem schimbarea de variabilă $t = \frac{\pi}{2} - x$ şi rezultă

$$I = \int_0^{\frac{\pi}{2}} \ln(\sin x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln(\cos x)\,\mathrm{d}x.$$

Atunci

$$\begin{aligned} 2I &= \int_0^{\frac{\pi}{2}} \ln(\sin x)\,\mathrm{d}x + \int_0^{\frac{\pi}{2}} \ln(\cos x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln(\sin x \cos x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln\left(\frac{\sin 2x}{2}\right)\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \ln(\sin 2x)\,\mathrm{d}x - \int_0^{\frac{\pi}{2}} \ln 2\,\mathrm{d}x \overset{2x=t}{=} \frac{1}{2}\int_0^{\pi} \ln\sin t\,\mathrm{d}t - \frac{\pi}{2}\ln 2 \\ &= \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \ln\sin t\,\mathrm{d}t + \int_{\frac{\pi}{2}}^{\pi} \ln\sin t\,\mathrm{d}t\right) - \frac{\pi}{2}\ln 2 \overset{u=\pi-t}{=} \frac{1}{2}(I+I) - \frac{\pi}{2}\ln 2. \end{aligned}$$

Rezultă $I = -\frac{\pi}{2}\ln 2$.

12.6. $\alpha = 1/2$ CONV Cu substituţia $x = \sin t$, obţinem

$$\int_0^1 \frac{\ln x}{\sqrt{1-x^2}}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln(\sin t)\,\mathrm{d}t = -\frac{\pi}{2}\ln 2.$$

12.7. $\alpha = 3/2$ CONV Cu substituţia $x = 1/t$ obţinem $I = -I$, adică $I = 0$.

12.8. $\alpha = 2$ CONV Cu substituţia $x = 1/t$ rezultă

$$I = \int_0^{\infty} \frac{\operatorname{arctg}\frac{1}{t}}{t^2+t+1}\,\mathrm{d}t.$$

Folosind faptul că $\operatorname{arctg} t + \operatorname{arctg}\frac{1}{t} = \frac{\pi}{2}$, pentru orice $t > 0$,

$$2I = \int_0^{\infty} \frac{\operatorname{arctg} t}{t^2+t+1}\,\mathrm{d}t + \int_0^{\infty} \frac{\operatorname{arctg}\frac{1}{t}}{t^2+t+1}\,\mathrm{d}t = \frac{\pi}{2}\int_0^{\infty} \frac{1}{t^2+t+1}\,\mathrm{d}t = \frac{\pi^2}{3\sqrt{3}}.$$

Rezultă $I = \frac{\pi^2}{6\sqrt{3}}$.

Probleme propuse

Să se studieze convergenţa integralelor:

Problema 12.9. $\displaystyle\int_1^{\infty} \frac{\sqrt{x+1}}{x\sqrt[3]{x^2+1}}\,\mathrm{d}x.$

Problema 12.10. $\displaystyle\int_0^{\infty} \frac{\operatorname{arctg} x}{x\sqrt{x}}\,\mathrm{d}x.$

Problema 12.11. $\displaystyle\int_1^{\infty} \frac{x}{(x-2)(x^2+1)}\,\mathrm{d}x.$

Problema 12.12. $\displaystyle\int_1^{e} \frac{1}{(x+1)\sqrt{\ln x}}\,\mathrm{d}x.$

Problema 12.13. $\displaystyle\int_0^1 \frac{e^{-\frac{1}{x}}}{x^3}\,\mathrm{d}x.$

Problema 12.14. $\displaystyle\int_1^\infty \frac{(\ln x)^4}{(x-1)^3}\,\mathrm{d}x.$

Problema 12.15. $\displaystyle\int_0^\infty \frac{\sin x}{x\sqrt{x}}\,\mathrm{d}x.$

Problema 12.16. $\displaystyle\int_2^\infty \frac{\cos x}{(x-1)^2}\,\mathrm{d}x.$

Problema 12.17. $\displaystyle\int_a^b \frac{1}{\sqrt{x-a}}\cdot\sin\frac{1}{(x-a)^2}\,\mathrm{d}x.$

Problema 12.18. $\displaystyle\int_a^b \frac{x^3}{\sqrt{(x-a)(b-x)}}\,\mathrm{d}x.$

Să se calculeze integralele:

Problema 12.19. $\displaystyle\int_0^\infty \frac{x^2}{(x+1)(x+2)^2(x+3)}\,\mathrm{d}x$

Problema 12.20. $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{(x^2+4)^n}.$

Problema 12.21. $\displaystyle\int_{-\infty}^\infty \frac{x^2+1}{(x^2+4)(x^2+9)}\,\mathrm{d}x.$

Problema 12.22. $\displaystyle\int_0^\infty \frac{x+1}{(x+2)^2(x+3)}\,\mathrm{d}x.$

Problema 12.23. $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{(x^2+1)\cdots(x^2+n)}.$

Problema 12.24. $\displaystyle\int_a^b \frac{\sqrt{x-a}}{\sqrt{b-x}}\,\mathrm{d}x.$

Problema 12.25. $\displaystyle\int_a^b \frac{x}{\sqrt{(x-a)(b-x)}}\,\mathrm{d}x.$

Problema 12.26. $\displaystyle\int_a^b \frac{x^2}{\sqrt{(x-a)(b-x)}}\,\mathrm{d}x.$

Problema 12.27. $\displaystyle\int_{-1}^1 \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x.$

Problema 12.28. $\displaystyle\int_0^1 \frac{1}{\sqrt{x-x^2}}\,\mathrm{d}x.$

Problema 12.29. $\displaystyle\int_0^\infty \frac{\ln x}{x^2+x+1}\,\mathrm{d}x.$

Problema 12.30. $\displaystyle\int_0^\infty \frac{\operatorname{arctg} x}{x^2+1}\,\mathrm{d}x.$

Problema 12.31. $\displaystyle\int_1^\infty \frac{\ln(1+x)}{x^3}\,\mathrm{d}x.$

Problema 12.32. $\displaystyle\int_0^\infty x^{2n+1}e^{-x^2}\,\mathrm{d}x.$