Seminar 1

Primitivele funcțiilor raționale

Toate integralele sunt de forma $\int \frac{P(x)}{Q(x)}\,\mathrm{d}x$, unde P, Q sunt polinoame cu coeficienți reali.

Să se calculeze integralele:

1.1. $\displaystyle\int_0^1 (x^4 - 4x^3 + 6x^2 - 5x - 2)\,\mathrm{d}x$

1.2. $\displaystyle\int_4^5 \frac{1}{(x-3)^6}\,\mathrm{d}x$

1.3. $\displaystyle\int_0^1 \frac{1}{x-2}\,\mathrm{d}x$

1.4. $\displaystyle\int_1^2 \frac{1}{2x-1}\,\mathrm{d}x$

1.5. $\displaystyle\int_0^{\sqrt{6}} \frac{x}{x^2+2}\,\mathrm{d}x$

1.6. $\displaystyle\int_0^3 \frac{1}{x^2+9}\,\mathrm{d}x$

1.7. $\displaystyle\int_0^1 \frac{1}{x^2+x+1}\,\mathrm{d}x$

1.8. $\displaystyle\int_0^1 \frac{x}{2x^2+x+1}\,\mathrm{d}x$

1.9. $\displaystyle\int \frac{x^5+x^4-8}{x^3-4x}\,\mathrm{d}x$

1.10. $\displaystyle\int_0^1 \frac{1}{(x+1)^2(x^2+1)}\,\mathrm{d}x$

1.11. $\displaystyle\int_0^2 \frac{1}{(x^2+4)^2}\,\mathrm{d}x$

Indicaţii şi răspunsuri

1.1. Se integrează cu formula $\int x^a \,\mathrm{d}x = \frac{x^{a+1}}{a+1} + C$, $a \neq -1$. Obţinem

$$I = \left(\frac{x^5}{5} - x^4 + 2x^3 - \frac{5x^2}{2} - 2x\right)\Bigg|_0^1 = \frac{1}{5} - 1 + 2 - \frac{5}{2} - 2 = -\frac{33}{10}.$$

1.2.

$$I = \int_4^5 \frac{\mathrm{d}x}{(x-3)^6} = \int_4^5 (x-3)^{-6}\,\mathrm{d}x = \frac{(x-3)^{-5}}{-5}\Bigg|_4^5 = -\frac{2^{-5}}{5} + \frac{1}{5} = \frac{31}{160}.$$

1.3. Se foloseşte formula generală $\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + C$. Rezultă $I = \ln|x-2|\Big|_0^1 = -\ln 2$.

1.4. Avem

$$I = \frac{1}{2}\int_1^2 \frac{2}{2x-1}\,\mathrm{d}x = \frac{1}{2}\cdot \ln|2x-1|\Bigg|_1^2 = \frac{\ln 3}{2}.$$

1.5.

$$I = \frac{1}{2}\int_0^{\sqrt{6}} \frac{2x}{x^2+4}\,\mathrm{d}x = \frac{1}{2}\cdot\ln(x^2+2)\Bigg|_0^{\sqrt{6}} = \frac{1}{2}\ln\frac{8}{2} = \frac{1}{2}\ln 2^2 = \ln 2.$$

1.6. Folosim formula $\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\operatorname{arctg}\frac{x}{a}$, $a \neq 0$. Obţinem

$$I = \frac{1}{\sqrt{3}}\cdot \operatorname{arctg}\frac{x}{3}\Bigg|_0^3 = \frac{1}{3}\cdot\operatorname{arctg} 1 = \frac{\pi}{12}.$$

1.7. Fiindcă expresia de gradul 2 de la numitor nu are rădăcini reale, facem un pătrat perfect $x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$. Atunci

$$I = \int_0^1 \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,\mathrm{d}x = \frac{2}{\sqrt{3}}\cdot\operatorname{arctg}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\Bigg|_0^1 = \frac{2}{\sqrt{3}}\cdot\operatorname{arctg}\frac{2x+1}{\sqrt{3}}\Bigg|_0^1 = \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}.$$

1.8. Fiindcă $(2x^2+x+1)' = 4x+1$, desfacem integrala în două

$$I = \frac{1}{4}\int_0^1 \frac{4x\,\mathrm{d}x}{2x^2+x+1} = \frac{1}{4}\int_0^1 \frac{4x+1\,\mathrm{d}x}{2x^2+x+1} - \frac{1}{4}\int_0^1 \frac{\mathrm{d}x}{2x^2+x+1}.$$

Pentru a doua integrală folosim forma canonică $ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]$. Obţinem

$$I = \frac{1}{4}\ln(2x^2+x+1)\Bigg|_0^1 - \frac{1}{8}\int_0^1 \frac{\mathrm{d}x}{\left(x+\frac{1}{4}\right)^2+\frac{7}{16}} = \frac{1}{4}\ln 4 - \frac{1}{2\sqrt{7}}\left(\operatorname{arctg}\frac{5}{\sqrt{7}} - \operatorname{arctg}\frac{1}{\sqrt{7}}\right).$$

1.9. Fiindcă polinomul de la numărător are gradul mai mare decât cel de la numitor, facem împărţirea de polinoame. Se obţine câtul x^2+x+4 şi restul $4x^2+16x-8$. Cu teorema împărţirii cu rest $D = \hat{I}\cdot C + R$, rezultă

$$x^5+x^4-8 = (x^3-4x)(x^2+x+4) + 4x^2+16x-8.$$

Aşadar,

$$\begin{aligned} I &= \int \frac{x^5+x^4-8}{x^3-4x}\,\mathrm{d}x = \int \frac{(x^3-4x)(x^2+x+4)+4x^2+16x-8}{x^3-4x}\,\mathrm{d}x \\ &= \int (x^2+x+4)\,\mathrm{d}x + \int \frac{4x^2+16x-8}{x^3-4x}\,\mathrm{d}x. \end{aligned}$$

Pentru că $x^3 - 4x = x(x^2-4) = x(x-2)(x+2)$ putem face descompunerea în fracţii simple

$$\frac{4x^2+16x-8}{x^3-4x} = \frac{4x^2+16x-8}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}.$$

Înmulţind cu numitorul comun $x(x-2)(x+2)$, rezultă

$$4x^2+16x-8 = A(x-2)(x+2) + Bx(x+2) + Cx(x-2).$$

Dând valoarea lui $x=0$, obţinem $A=2$. Pentru $x=2$, avem $B=5$, iar pentru $x=-2$ se obţine $C=-3$. Deci

$$\int \frac{4x^2+16x-8}{x^3-4x}\,\mathrm{d}x = \int \frac{2}{x}\,\mathrm{d}x + \int \frac{5}{x-2}\,\mathrm{d}x + \int \frac{-3}{x+2}\,\mathrm{d}x.$$

În final,

$$I = \frac{x^3}{3} + \frac{x^2}{2} + 4x + 2\ln|x| + 5\ln|x-2| - 3\ln|x+2| + C.$$

1.10. Expresia de sub integrală se descompune

$$\frac{1}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1},$$

cu $A=1/2$, $B=1/2$, $C=-1/2$, $D=0$. Integrala va fi

$$I = \left(\frac{1}{2}\ln|x+1| - \frac{1}{2(x+1)} - \frac{1}{4}\ln(x^2+1)\right)\Bigg|_0^1 = \frac{1+\ln 2}{4}.$$

1.11. Scriem

$$\begin{aligned}
\int_0^2 \frac{\mathrm{d}x}{(x^2+4)^2} &= \frac{1}{4}\int_0^2 \frac{4\,\mathrm{d}x}{(x^2+4)^2} \\
&= \frac{1}{4}\int_0^2 \frac{x^2+4-x^2}{(x^2+4)^2}\,\mathrm{d}x \\
&= \frac{1}{4}\int_0^2 \frac{\mathrm{d}x}{x^2+4} - \frac{1}{4}\int_0^2 \frac{x^2}{(x^2+4)^2}\,\mathrm{d}x \\
&= \frac{1}{8}\operatorname{arctg}\frac{x}{2}\Bigg|_0^2 - \frac{1}{4}\int_0^2 \frac{x^2}{(x^2+4)^2}\,\mathrm{d}x.
\end{aligned}$$

Folosim integrarea prin părţi. Avem

$$\begin{aligned}
\int_0^2 \frac{x^2}{(x^2+4)^2}\,\mathrm{d}x &= \frac{1}{2}\int_0^2 x\cdot\frac{2x}{(x^2+4)^2}\,\mathrm{d}x = \frac{1}{2}\int_0^2 x\cdot\left(\frac{-1}{x^2+4}\right)'\,\mathrm{d}x \\
&= -\frac{x}{2(x^2+4)}\Bigg|_0^2 - \frac{1}{2}\int_0^2 x'\cdot\frac{-1}{x^2+4}\,\mathrm{d}x = -\frac{1}{8} + \frac{1}{4}\operatorname{arctg}\frac{x}{2}\Bigg|_0^2 = -\frac{1}{8} + \frac{\pi}{16}.
\end{aligned}$$

Atunci

$$I = \frac{1}{32} + \frac{\pi}{64}.$$

Probleme suplimentare

1.12. $\displaystyle\int_0^1 \frac{x}{x^4+1}\,\mathrm{d}x$

1.13. $\displaystyle\int_0^1 \frac{x^3}{x^4+1}\,\mathrm{d}x$

1.14. $\displaystyle\int_1^2 \frac{x^2+1}{x^4+1}\,\mathrm{d}x$

1.15. $\displaystyle\int_1^2 \frac{x^2-1}{x^4+1}\,\mathrm{d}x$

1.16. $\displaystyle\int_1^2 \frac{1}{x^4+1}\,\mathrm{d}x$

1.17. $\displaystyle\int_1^2 \frac{x^2}{x^4+1}\,\mathrm{d}x$

1.18. $\displaystyle\int_1^2 \frac{3x^2+2}{x^4+1}\,\mathrm{d}x$

1.19. $\displaystyle\int_0^1 \frac{x^4}{x^2-x+1}\,\mathrm{d}x$

1.20. $\displaystyle\int_0^1 \frac{2x-1}{(x^2+1)(2x+1)}\,\mathrm{d}x$

1.21. $\displaystyle\int_0^1 \frac{1}{(x^2+1)(x^2+2)(x^2+3)}\,\mathrm{d}x$

1.22. $\displaystyle\int_0^1 \frac{x-x^2}{(x^2+1)(x^3+1)}\,\mathrm{d}x$

1.23. $\displaystyle\int_{\frac{1}{3}}^3 \frac{1}{(x^2+1)(x^5+1)}\,\mathrm{d}x$

1.24. $\displaystyle\int_{\frac{1}{2}}^2 \frac{x^3}{1+x+x^2+x^3+x^4+x^5}\,\mathrm{d}x$

1.25. $\displaystyle\int_0^1 \frac{1}{(x+1)^2(x+2)^5}\,\mathrm{d}x$