Seminar 12

Integrale duble

Să se calculeze

12.1. $\displaystyle\iint_D 4x\,\mathrm{d}x\,\mathrm{d}y$, unde D este interiorul triunghiului ABC, $A(1,1)$, $B(3,3)$ şi $C(1,4)$.

12.2. $\displaystyle\iint_D x^{p-1}y^{q-1}\,\mathrm{d}x\,\mathrm{d}y$, unde $p,q\in\mathbb{N}$, iar $D=\{\,(x,y)\,|\,x+y\le 1, x,y\ge 0\,\}$.

12.3. $\displaystyle\iint_D e^{\frac{x}{y}}\,\mathrm{d}x\,\mathrm{d}y$, unde D este limitat de $x=y^2$, $x=0$ şi $y=1$.

12.4. $\displaystyle\iint_D \frac{\ln(x^2+y^2)}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y$, unde D este $1\le x^2+y^2\le e^2$.

12.5. $\displaystyle\iint_D x\,\mathrm{d}x\,\mathrm{d}y$, unde $D=\{\,(x,y)\,|\,x^2+y^2\le 2,\ y\ge x\,\}$.

12.6. Aria domeniului mărginit de elipsa $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Indicaţii

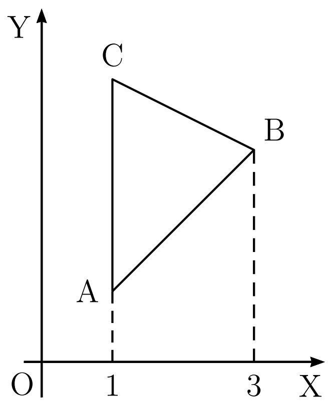


Figura 12.1: Triunghiul ABC din problema 1

12.1. Ecuaţia dreptei AB este $y=x$ şi ecuaţia dreptei BC : $y=(9-x)/2$. Scriem domeniul sub forma

$$D=\left\{\,(x,y)\,|\,x\in[1,3],\ x\le y\le\frac{9-x}{2}\,\right\}.$$

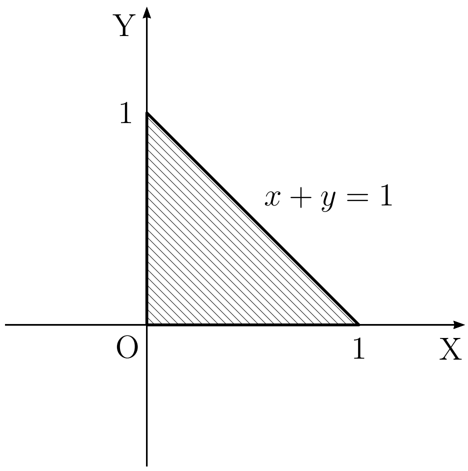


Figura 12.2: Domeniul D din problema 2

Atunci

$$I = \int_1^3 \left(\int_x^{\frac{9-x}{2}} 4x \,\mathrm{d}y \right) \mathrm{d}x$$
$$= \int_1^3 4x \left(\frac{9-x}{2} - x \right) \mathrm{d}x = \int_1^3 (18x - 6x^2) \,\mathrm{d}x = 20.$$

12.2.

$$I = \int_0^1 \left(\int_0^{1-x} x^{p-1} y^{q-1} \,\mathrm{d}y \right) \mathrm{d}x = \frac{1}{q} \int_0^1 x^{p-1} (1-x)^q \,\mathrm{d}x$$
$$= \frac{1}{q} \beta(p, q+1) = \frac{(p-1)!(q-1)!}{(p+q)!}.$$

12.3. Domeniul D este reprezentat în Figura 12.3.

$$\iint_D e^{\frac{x}{y}} \,\mathrm{d}x \,\mathrm{d}y = \int_0^1 \mathrm{d}y \int_0^{y^2} e^{\frac{x}{y}} \,\mathrm{d}x = \int_0^1 \mathrm{d}y \left[y \, e^{\frac{x}{y}} \Big|_0^{y^2} \right]$$
$$= \int_0^1 y \left(e^y - 1 \right) \mathrm{d}y = \int_0^1 y e^y \,\mathrm{d}y - \int_0^1 y \,\mathrm{d}y$$
$$= y \, e^y \Big|_0^1 - e^y \Big|_0^1 - \frac{1}{2} y^2 \Big|_0^1 = \frac{1}{2}.$$

12.4. Se face schimbarea de variabile la coordonate polare $x = \rho \cos \varphi$, $y = \rho \sin \varphi$. Noul domeniu este $(\rho, \varphi) \in \Delta = [1, e] \times [0, 2\pi]$, iar $|J| = \rho$. Valoarea integralei este

$$\iint_\Delta \frac{\ln \rho^2}{\rho^2} \cdot |J| \,\mathrm{d}\rho \,\mathrm{d}\varphi = \left(\int_1^e \frac{2 \ln \rho}{\rho} \,\mathrm{d}\rho \right) \cdot \left(\int_0^{2\pi} \mathrm{d}\varphi \right) = 2\pi \ln^2 \rho \Big|_1^e = 2\pi.$$

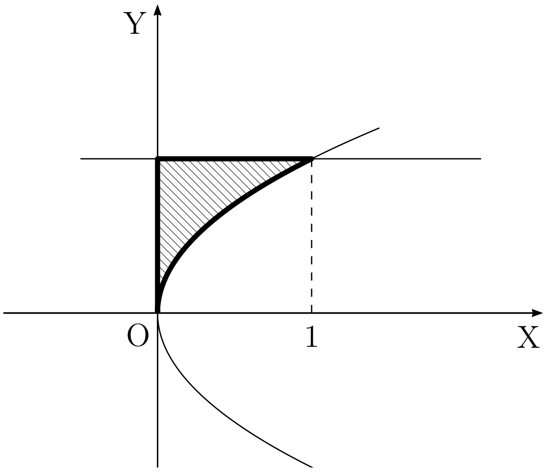


Figura 12.3: Problema 12.3

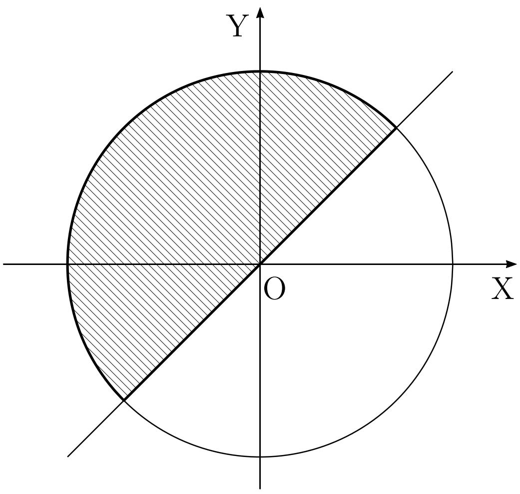


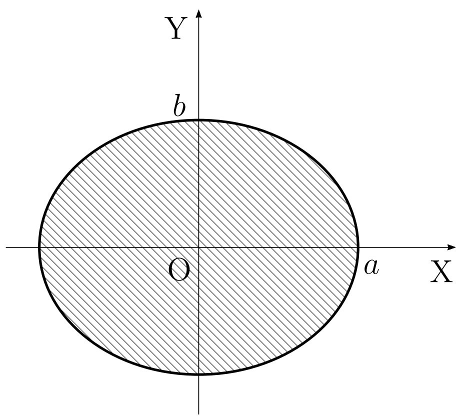
Figura 12.4: Domeniul de la problema 6

12.5. Se face schimbarea de variabile la coordonate polare $x = \rho\cos\varphi$, $y = \rho\sin\varphi$. Noul domeniu este $(\rho,\varphi) \in \Delta = [0,\sqrt{2}] \times [\frac{\pi}{4}, \pi + \frac{\pi}{4}]$, iar $|J| = \rho$. Integrala se calculează

$$\iint_D x \,\mathrm{d}x\,\mathrm{d}y = \iint_\Delta \rho\cos\varphi |J| \,\mathrm{d}\rho\,\mathrm{d}\varphi = \left(\int_0^{\sqrt{2}} \rho^2 \,\mathrm{d}\rho\right) \cdot \left(\int_{\frac{\pi}{4}}^{\pi+\frac{\pi}{4}} \cos\varphi \,\mathrm{d}\varphi\right) = \frac{2\sqrt{2}}{3} \cdot (-\sqrt{2}) = -\frac{4}{3}.$$

12.6. Se face schimbarea de variabile la coordonate polare generalizate $x = a\rho\cos\varphi$, $y = b\rho\sin\varphi$. Noul domeniu este $(\rho,\varphi) \in \Delta = [0,1] \times [0,2\pi]$, iar $|J| = ab\rho$. Aria va fi

$$Aria(\text{elipsei}) = \iint_D \mathrm{d}x\,\mathrm{d}y = \iint_\Delta |J| \,\mathrm{d}\rho\,\mathrm{d}\varphi = ab\left(\int_0^1 \rho \,\mathrm{d}\rho\right) \cdot \left(\int_0^{2\pi} \mathrm{d}\varphi\right) = \pi ab.$$


Y
b
O
a
X