Seminar 13

Integrale triple

Să se calculeze

13.1. $\iiint_V (1+x+y+z)^{-3}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, unde

$$V = \{\,(x,y,z)\,|\,x+y+z \le 1,\ x,y,z \ge 0\,\}.$$

13.2. $\iiint_V (x^2+y^2)^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, unde V este corpul $x^2+y^2 \le z^2$, $0 \le z \le 1$.

13.3. $\iiint_V z \cdot e^{-(x^2+y^2+z^2)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, unde

$$V = \left\{\,(x,y,z)\,|\,x^2+y^2+z^2 \le 4,\ z \ge 0\,\right\}.$$

13.4. $\iiint_V \dfrac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{x^2+y^2+z^2}$, unde V este mulţimea punctelor care verifică

$$1 \le x^2+y^2+z^2 \le 4,\ x^2+y^2 \le 3z^2,\ z \ge 0.$$

Temă

13.5. Să se calculeze

$$\iiint_V (2z-y^2+xz)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

unde V este paralelipipedul dreptunghic $[-1,1]\times[1,2]\times[0,3]$.

13.6. Volumul corpului mărginit de paraboloidul $x^2+y^2=z$ şi planul $z=2$.

13.7. $\iiint_V \sqrt{x^2+y^2+z^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, unde

$$V = \left\{\,(x,y,z)\,|\,x^2+y^2+z^2 \le 9,\ x,y,z \ge 0\,\right\}.$$

13.8. $\iiint_V \sqrt{1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, unde V este $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1$.

Indicaţii

13.1. Scriem corpul V sub forma

$$V = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in D,\ g_1(x, y) \leq z \leq g_2(x, y)\}$$

şi aplicăm formula

$$\iiint_V f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \iint_D \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)\, \mathrm{d}z \right] \mathrm{d}x\, \mathrm{d}y.$$

Proiecţia corpului V

$$V = \{\, (x, y, z) \mid x + y + z \leq 1,\ x, y, z \geq 0 \,\}$$

pe planul XOY este triunghiul format de dreptele $x = 0$, $y = 0$ şi $x + y = 1$. Aşadar

$$D = \left\{\, (x, y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ x + y \leq 1 \,\right\}.$$

Dacă fixăm un punct (x, y) din D, atunci pentru ca $(x, y, z) \in V$, z ia valori de la planul XOY

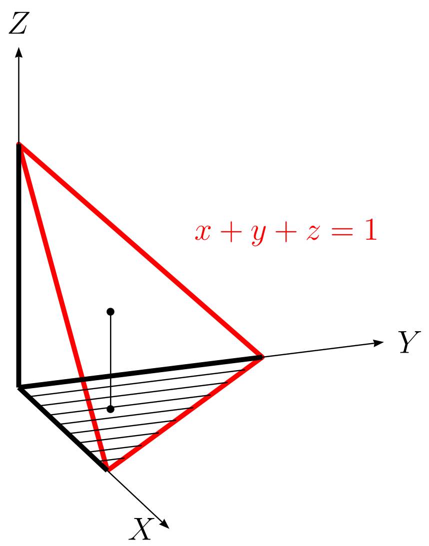


Figura 13.1: Tetraedrul V

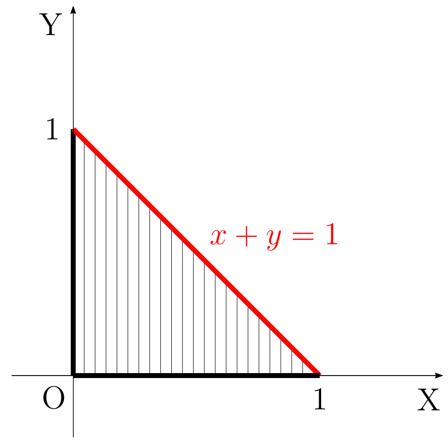


Figura 13.2: Domeniul D

la planul $x + y + z = 1$. Deci când $(x, y) \in D$ avem $0 \leq z \leq 1 - x - y$. Rezultă

$$\begin{aligned}
\iiint_V \frac{\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z}{(1 + x + y + z)^3} &= \iint_D \mathrm{d}x\, \mathrm{d}y \int_0^{1-x-y} \frac{\mathrm{d}z}{(1 + x + y + z)^3} \\
&= \iint_D \left(\frac{1}{2(1 + x + y)^2} - \frac{1}{8} \right) \mathrm{d}x\, \mathrm{d}y \\
&= \frac{1}{2} \iint_D \frac{1}{(1 + x + y)^2}\, \mathrm{d}x\, \mathrm{d}y - \frac{1}{8} \iint_D \mathrm{d}x\, \mathrm{d}y \\
&= \frac{1}{2} \int_0^1 \mathrm{d}x \int_0^{1-x} \frac{\mathrm{d}y}{(1 + x + y)^2} - \frac{1}{8} \mathcal{A}_D \\
&= \frac{1}{2} \int_0^1 \left(\frac{1}{1 + x} - \frac{1}{2} \right) \mathrm{d}x - \frac{1}{8} \cdot \frac{1}{2} = \frac{1}{2} \ln 2 - \frac{5}{16}.
\end{aligned}$$

13.2. Corpul V se scrie sub forma

$$V = \left\{ (x,y,z) \,|\, (x,y) \in D,\ \sqrt{x^2+y^2} \le z \le 1 \right\},$$

unde D este discul $x^2 + y^2 \le 1$. Valoarea integralei se calculează astfel

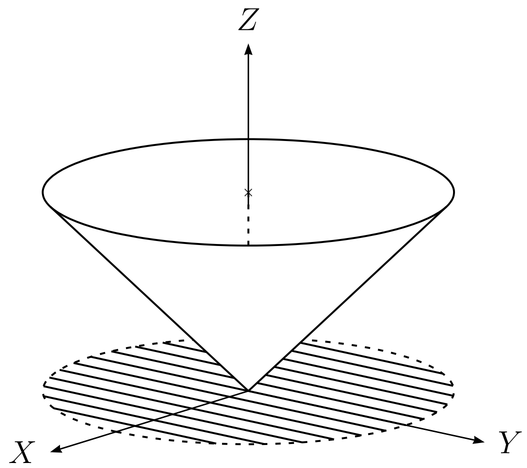


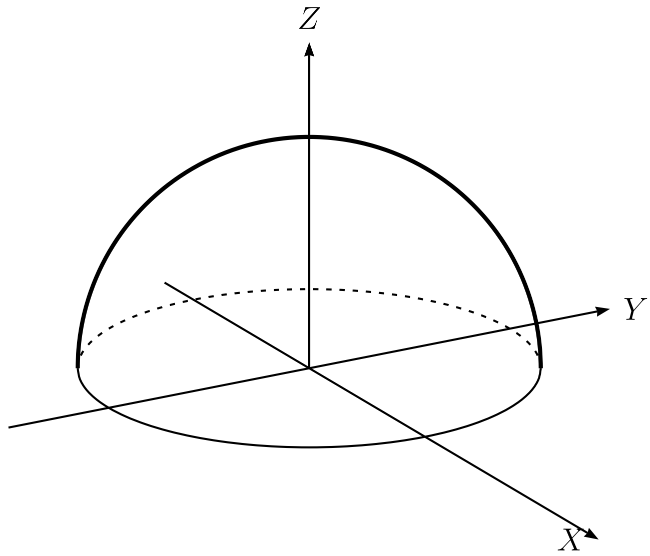
Figura 13.3: Conul $x^2 + y^2 = z^2$

$$\begin{aligned} I = \iiint_V (x^2+y^2)^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \iint_D \left(\int_{\sqrt{x^2+y^2}}^1 (x^2+y^2)^2 \,\mathrm{d}z \right) \mathrm{d}x\,\mathrm{d}y \\ &= \iint_D (x^2+y^2)^2 \left[1 - \sqrt{x^2+y^2} \right] \mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Facem schimbarea de variabile la coordonate polare $x = \rho\cos\varphi$, $y = \rho\sin\varphi$ şi obţinem

$$\begin{aligned} I = \iint_D (x^2+y^2)^2 \left[1 - \sqrt{x^2+y^2} \right] \mathrm{d}x\,\mathrm{d}y &= \int_0^{2\pi} \int_0^1 \rho^4(1-\rho)\cdot\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi \\ &= 2\pi \int_0^1 (\rho^5 - \rho^6)\,\mathrm{d}\rho = 2\pi \left(\frac{1}{6} - \frac{1}{7} \right) = \frac{\pi}{21}. \end{aligned}$$

13.3. Ecuaţia $x^2+y^2+z^2=4$ reprezintă o sferă de centru $(0,0,0)$ şi rază 2. Corpul V



reprezintă jumătatea superioară a unei bile de rază 2. Trecem la coordonate sferice

$$\begin{cases} x = \rho\cos\varphi\sin\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\theta. \end{cases}$$

Înlocuind aceste relaţii în inegalitatea $x^2+y^2+z^2 \le 4$ obţinem $\rho \in [0,2]$. Din $z \ge 0$ obţinem $\theta \in \left[0, \frac{\pi}{2}\right]$. Asupra lui φ nu este nicio restricţie, deci $\varphi \in [0, 2\pi]$. Jacobianul la coordonate sferice este $|J| = \rho^2\sin\theta$. Cu acestea valoarea integralei este

$$I = \iiint_V z \cdot e^{-(x^2+y^2+z^2)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^2 \left[\int_0^{\frac{\pi}{2}} \left(\int_0^{2\pi} \rho\cos\theta \cdot e^{-\rho^2}\rho^2\sin\theta\,\mathrm{d}\varphi\right)\mathrm{d}\theta\right]\mathrm{d}\rho$$
$$= 2\pi\left(\int_0^2 \rho^3 e^{-\rho^2}\,\mathrm{d}\rho\right)\cdot\left(\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\,\mathrm{d}\theta\right).$$

Avem

$$\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \sin\theta(\sin\theta)'\,\mathrm{d}\theta = \left.\frac{\sin^2\theta}{2}\right|_0^{\frac{\pi}{2}} = \frac{1}{2}$$

şi prin schimbarea de variabilă $-\rho^2 = t$

$$\int_0^2 \rho^3 e^{-\rho^2}\,\mathrm{d}\rho = \frac{1}{2}\int_0^2 e^{-\rho^2}(-\rho^2)(-2\rho\,\mathrm{d}\rho) = \frac{1}{2}\int_0^{-4} e^t t\,\mathrm{d}t = -\frac{4e^{-4}}{2} - \frac{1}{2}\int_0^{-4} e^t\,\mathrm{d}t$$
$$= -\frac{4e^{-4}}{2} - \frac{e^{-4}}{2} + \frac{1}{2} = \frac{1}{2}\left(1-5e^{-4}\right).$$

Rezultă

$$I = \frac{\pi}{2}\left(1-5e^{-4}\right).$$

13.4. Mulţimea V este

$$V = \left\{ (x,y,z) \in \mathbb{R}^3 \,|\, 1 \leq x^2+y^2+z^2 \leq 4,\ x^2+y^2 \leq 3z^2,\ z \geq 0 \right\}.$$

Trecem la coordonate sferice. Înlocuim pe $x = \rho\cos\varphi\sin\theta$, $y = \rho\sin\varphi\sin\theta$ şi $z = \rho\cos\theta$ în

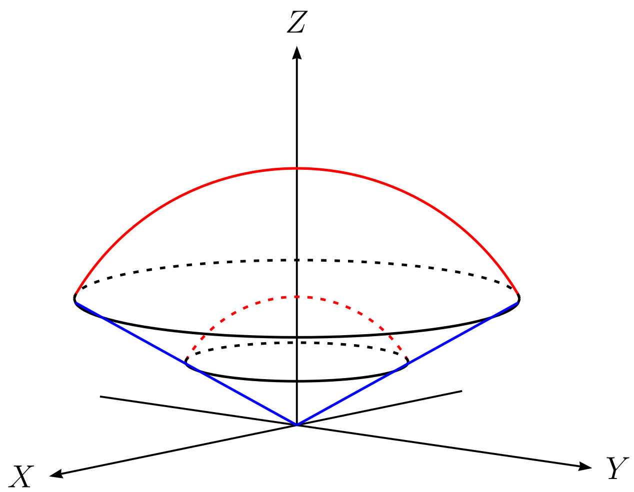


Figura 13.4: Domeniul spaţial cuprins între cele 2 sfere şi în interiorul conului

condiţiile care definesc pe V. Obţinem

$$1 \leq x^2+y^2+z^2 = \rho^2 \leq 4 \quad \text{şi} \quad x^2+y^2 = \rho^2\sin^2\theta \leq 3\rho^2\cos^2\theta \quad \text{şi} \quad z = \rho\cos\theta \geq 0$$

Aşadar, $\operatorname{tg}^2\theta \leq 3$ şi $\cos\theta \geq 0$

$$\Omega = \left\{ (\rho,\theta,\varphi) \,|\, 1 \leq \rho \leq 2,\ 0 \leq \theta \leq \frac{\pi}{3},\ 0 \leq \varphi \leq 2\pi \right\}.$$

Calculăm integrala dată în felul următor:

$$\begin{aligned}\iiint_V \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{x^2+y^2+z^2} &= \iiint_\Omega \frac{|J|\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta}{\rho^2} = \int_1^2 \mathrm{d}\rho \int_0^{\frac{\pi}{3}} \mathrm{d}\theta \int_0^{2\pi} \frac{\rho^2\sin\theta\,\mathrm{d}\varphi}{\rho^2} \\ &= 2\pi \int_1^2 \mathrm{d}\rho \int_0^{\frac{\pi}{3}} \sin\theta\,\mathrm{d}\theta = 2\pi \int_1^2 (-\cos\theta)\Big|_0^{\frac{\pi}{3}} \mathrm{d}\rho = \pi.\end{aligned}$$