Seminar 2

Primitive de forma $\int R(\sin x, \cos x)\,\mathrm{d}x$

Funcţiile R sunt funcţii raţionale de forma $R(x) = \frac{P(x)}{Q(x)}$, unde P, Q sunt polinoame cu coeficienți reali.

Să se calculeze integralele:

2.1. $\displaystyle\int_0^{\frac{\pi}{3}} (\cos 5x - \sin 2x)\,\mathrm{d}x$

2.2. $\displaystyle\int_0^{\frac{\pi}{4}} \cos 2x \cdot \sin 3x\,\mathrm{d}x$

2.3. $\displaystyle\int_0^{\frac{\pi}{2}} \sin 2x \cdot \sin 3x\,\mathrm{d}x$

2.4. $\displaystyle\int_0^{\frac{\pi}{4}} \cos x \cdot \cos 3x\,\mathrm{d}x$

2.5. $\displaystyle\int_0^{\pi} \cos^4 3x\,\mathrm{d}x$

2.6. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^3 x\,\mathrm{d}x$

2.7. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{2 + \cos x}\,\mathrm{d}x$

2.8. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{(2 + \sin x)(1 + \sin x)}\,\mathrm{d}x$

2.9. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos^6 x}\,\mathrm{d}x$

2.10. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{\sin^4 x + \cos^4 x}\,\mathrm{d}x$

2.11. $\displaystyle\int_0^{\pi} \frac{1}{5 + 4\sin x}\,\mathrm{d}x$

Indicaţii şi răspunsuri

2.1. Se folosesc formulele

$$\int \sin ax \,\mathrm{d}x = -\frac{\cos ax}{a}, \qquad \int \cos ax \,\mathrm{d}x = \frac{\sin ax}{a}, \quad a \neq 0.$$

Obţinem

$$I = \int_0^{\frac{\pi}{3}} (\cos 5x - \sin 2x)\,\mathrm{d}x = \int_0^{\frac{\pi}{3}} \cos 5x \,\mathrm{d}x - \int_0^{\frac{\pi}{3}} \sin 2x \,\mathrm{d}x = \frac{1}{5}\sin 5x\Big|_0^{\frac{\pi}{3}} + \frac{1}{2}\cos 2x\Big|_0^{\frac{\pi}{3}} = -\frac{\sqrt{3}}{10} - \frac{3}{4}.$$

2.2. Folosim formula

$$\sin a \cdot \cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$$

pentru $a = 3x$ şi $b = 2x$. Se obţine

$$I = \left(-\frac{1}{10}\cos 5x - \frac{1}{2}\cos x\right)\Big|_0^{\frac{\pi}{4}} = -\frac{1}{10}\cos\frac{5\pi}{4} + \frac{1}{10} - \frac{\sqrt{2}}{4} + \frac{1}{2}.$$

Fiindcă $\cos\frac{5\pi}{4} = \cos\left(\pi + \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ se obţine $I = \frac{6-4\sqrt{2}}{10} = \frac{3-\sqrt{2}}{5}$.

2.3. Se foloseşte formula

$$\sin a \cdot \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

şi se obţine

$$I = \frac{1}{2}\sin x\Big|_0^{\frac{\pi}{2}} - \frac{1}{10}\sin 5x\Big|_0^{\frac{\pi}{2}} = \frac{1}{2} - \frac{1}{10} = \frac{2}{5}.$$

2.4. Cu ajutorul formulei

$$\cos a \cdot \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$

şi se obţine

$$I = \frac{1}{4}\sin 2x\Big|_0^{\frac{\pi}{4}} + \frac{1}{8}\sin 4x\Big|_0^{\frac{\pi}{4}} = \frac{1}{4}.$$

2.5. Cu $a = b$ în formulele din exercițiul anterior, rezultă noile formule

$$\cos^2 a = \frac{1+\cos 2a}{2}, \qquad \sin^2 a = \frac{1-\cos 2a}{2}.$$

Vom obţine

$$I = \int_0^{\pi} \cos^4 3x \,\mathrm{d}x = \frac{1}{4}\int_0^{\pi} (1+\cos 6x)^2 \,\mathrm{d}x = \frac{\pi}{4} + \frac{1}{2}\int_0^{\pi} \cos 6x \,\mathrm{d}x + \frac{1}{4}\int_0^{\pi} \cos^2 6x \,\mathrm{d}x.$$

Aplicând încă o dată formula de liniarizare, rezultă

$$I = \frac{\pi}{4} + \frac{\sin 6x}{12}\Big|_0^{\pi} + \frac{1}{8}\int_0^{\pi} (1+\cos 12x)\,\mathrm{d}x = \frac{\pi}{4} + 0 + \frac{\pi}{8} + 0 = \frac{3\pi}{8}.$$

2.6. Scriem $\sin^3 x = \sin^2 x \cdot \sin x = (1-\cos^2 x)\sin x$ şi facem schimbarea de variabilă $\cos x = t$. Obţinem

$$I = \int_1^0 (1-t^2)(-\,\mathrm{d}t) = \left(t - \frac{t^3}{3}\right)\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

2.7. Pentru integrale de forma $\int R(\sin x, \cos x)\,\mathrm{d}x$, dacă se verifică condiţia

$$R(-\sin x, \cos x) = -R(\sin x, \cos x)$$

atunci se face schimbarea de variabilă $u = \cos x$. Avem

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{2+\cos x}\,\mathrm{d}x = \int_1^0 \frac{(1-u^2)(-\,\mathrm{d}u)}{2+u} = -\int_0^1 \frac{u^2-1}{u+2}\,\mathrm{d}u = -\int_0^1 \frac{u^2-4+3}{u+2}\,\mathrm{d}u$$
$$= -\int_0^1 \left(u - 2 + \frac{3}{u+2}\right)\mathrm{d}u = -\left(\frac{u^2}{2} - 2u + 3\ln|u+2|\right)\Big|_0^1 = \frac{3}{2} + 3\ln 2 - 3\ln 3.$$

2.8. Pentru integrale de forma $\int R(\sin x, \cos x)\,\mathrm{d}x$, dacă se verifică condiţia

$$R(\sin x, -\cos x) = -R(\sin x, \cos x)$$

atunci se face schimbarea de variabilă $u = \sin x$. În cazul nostru, cu această schimbare de variabilă obţinem

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{(2+\sin x)(1+\sin x)}\,\mathrm{d}x = \int_0^1 \frac{1}{(2+u)(1+u)}\,\mathrm{d}u = \int_0^1 \frac{(2+u)-(1+u)}{(2+u)(1+u)}\,\mathrm{d}u$$
$$= \int_0^1 \left[\frac{1}{1+u} - \frac{1}{2+u}\right]\mathrm{d}u = \left[\ln(u+1) - \ln(u+2)\right]\Big|_0^1 = 2\ln 2 - \ln 3.$$

2.9. Atunci când este îndeplinită condiţia $R(-\sin x, -\cos x) = R(\sin x, \cos x)$ se face substituţia $u = \operatorname{tg} x$ şi se ţine cont că $\mathrm{d}x = \frac{\mathrm{d}u}{1+u^2}$, $\sin x = \frac{u}{\sqrt{1+u^2}}$ şi $\cos x = \frac{1}{\sqrt{1+u^2}}$.
Avem

$$I = \int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos^6 x}\,\mathrm{d}x = \int_0^1 \frac{\frac{u^2}{1+u^2}}{\frac{1}{(1+u^2)^3}} \frac{\mathrm{d}u}{1+u^2} = \int_0^1 u^2(1+u^2)\,\mathrm{d}u = \int_0^1 (u^2+u^4)\,\mathrm{d}u = \frac{1}{3} + \frac{1}{5} = \frac{8}{15}.$$

2.10.

$$I = \int_0^{\frac{\pi}{4}} \frac{\mathrm{d}x}{\cos^4 x + \sin^4 x} = \int_0^1 \frac{(1+u^2)\,\mathrm{d}u}{1+u^4} = \int_0^1 \frac{\left(1+\frac{1}{u^2}\right)\mathrm{d}u}{u^2+\frac{1}{u^2}}.$$

Cu schimbarea de variabilă $t = u - \frac{1}{u}$ avem $\mathrm{d}t = \left(1+\frac{1}{u^2}\right)\mathrm{d}u$ şi $t^2 = u^2 + \frac{1}{u^2} - 2$. Aşadar

$$I = \int_{-\infty}^0 \frac{\mathrm{d}t}{t^2+2} = \frac{1}{\sqrt{2}} \operatorname{arctg} \frac{t}{\sqrt{2}}\Big|_{-\infty}^0 = \frac{\pi}{2\sqrt{2}}.$$

d)

$$I = \int \frac{\mathrm{d}x}{\sin^2 x \cos^4 x} = \int \frac{(1+u^2)^2\,\mathrm{d}u}{u^2} = -\frac{1}{u} + 2u + \frac{u^3}{3} = -\operatorname{ctg} x + 2\operatorname{tg} x + \frac{\operatorname{tg}^3 x}{3} + C.$$

2.11. În cazul în care nici o condiţie de la exerciţiile anterioare nu este verificată se face substituţia $u = \operatorname{tg}\frac{x}{2}$ şi se ţine cont de faptul că $\mathrm{d}x = \frac{2\,\mathrm{d}u}{1+u^2}$, $\sin x = \frac{2u}{1+u^2}$ şi $\cos x = \frac{1-u^2}{1+u^2}$. Avem

$$I = \int_0^{\pi} \frac{\mathrm{d}x}{5+4\sin x} = 2\int_0^{\infty} \frac{\mathrm{d}u}{5+5u^2+8u} = \frac{2}{5}\int_0^{\infty} \frac{\mathrm{d}u}{\left(u+\frac{4}{5}\right)^2 + \frac{9}{25}} = \frac{2}{3}\operatorname{arctg}\frac{4+5u}{3}\Big|_0^{\infty} = \frac{\pi}{3} - \frac{2}{3}\operatorname{arctg}\frac{4}{3}.$$

Probleme suplimentare

2.12. $\displaystyle\int_0^{\frac{\pi}{2}} \cos^5 x\,\mathrm{d}x.$

2.13. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos x+\cos^2 x}\,\mathrm{d}x$

2.14. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin^4 x}\,\mathrm{d}x$

2.15. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{(\sin x+\cos x)^2}\,\mathrm{d}x$

2.16. $\displaystyle\int_0^{\frac{\pi}{2}} \sin(3x)\cos^2 x\,\mathrm{d}x.$

2.17. $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin x}{\sin^2 x+\cos x+1}\,\mathrm{d}x.$

2.18. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin^4 x}{\sin^6 x+1}\,\mathrm{d}x.$

2.19. $\displaystyle\int_0^{\pi} \frac{1}{2-\sin x}\,\mathrm{d}x.$

2.20. $\displaystyle\int_0^{4\pi} \frac{1}{5-3\cos x}\,\mathrm{d}x.$