Seminar 2

Primitive de forma $\int R(\sin x, \cos x)\,\mathrm{d}x$

Funcţiile R sunt funcţii raţionale de forma $R(x) = \frac{P(x)}{Q(x)}$, unde P, Q sunt polinoame cu coeficienți reali.

Să se calculeze integralele:

Problema 2.1.

a) $\displaystyle\int (\cos 5x - \sin 2x)\,\mathrm{d}x$

b) $\displaystyle\int \cos 2x \cdot \sin 3x\,\mathrm{d}x$

c) $\displaystyle\int \sin 3x \cdot \sin 2x\,\mathrm{d}x$

d) $\displaystyle\int \cos x \cdot \cos 3x\,\mathrm{d}x$

Problema 2.2.

a) $\displaystyle\int \cos^2 x\,\mathrm{d}x,$

b) $\displaystyle\int \sin^2 3x\,\mathrm{d}x$

c) $\displaystyle\int \sin^4 x\,\mathrm{d}x$

d) $\displaystyle\int \cos^5 x\,\mathrm{d}x$

Problema 2.3.

a) $\displaystyle\int \frac{\sin^3 x}{2 + \cos x}\,\mathrm{d}x$

b) $\displaystyle\int \frac{\sin x}{1 + \cos x + \cos^2 x}\,\mathrm{d}x$

c) $\displaystyle\int \frac{\sin x\,\mathrm{d}x}{\cos 2x}$

d) $\displaystyle\int \frac{\mathrm{d}x}{\sin x + \sin 2x}$

Problema 2.4.

a) $\displaystyle\int \frac{\cos x}{\sin x}\,\mathrm{d}x$

b) $\displaystyle\int \frac{\cos^3 x}{\sin^4 x}\,\mathrm{d}x$

c) $\displaystyle\int \frac{\mathrm{d}x}{\cos x \cos 2x}$

d) $\displaystyle\int \frac{\sin 2x\,\mathrm{d}x}{(2 + \sin x)^2}$

Problema 2.5.

a) $\displaystyle\int \frac{\sin^2 x}{\cos^6 x}\,\mathrm{d}x$

b) $\displaystyle\int \frac{\mathrm{d}x}{(\sin x + \cos x)^2}$

c) $\displaystyle\int \frac{\mathrm{d}x}{\cos^4 x + \sin^4 x}$

d) $\displaystyle\int \frac{\mathrm{d}x}{\sin^2 x \cos^4 x}$

Problema 2.6.

a) $\displaystyle\int \frac{\mathrm{d}x}{5 + 4\sin x}$

b) $\displaystyle\int \frac{\mathrm{d}x}{5 - 3\cos x}$

Indicaţii şi răspunsuri

2.1. a) Se folosesc formulele

$$\int \sin ax\,\mathrm{d}x = -\frac{\cos ax}{a}, \qquad \int \cos ax\,\mathrm{d}x = \frac{\sin ax}{a}, \quad a \neq 0.$$

Obţinem

$$I = \int (\cos 5x - \sin 2x)\,\mathrm{d}x = \int \cos 5x\,\mathrm{d}x - \int \sin 2x\,\mathrm{d}x = \frac{1}{5}\sin 5x + \frac{1}{2}\cos 2x + C.$$

b) Folosim formula

$$\sin a \cdot \cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$$

pentru $a = 3x$ şi $b = 2x$. Se obţine $I = -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + C$.
c) Se foloseşte formula

$$\sin a \cdot \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

şi se obţine $I = \frac{1}{2}\sin x - \frac{1}{10}\sin 5x + C$.
d) Cu ajutorul formulei

$$\cos a \cdot \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$

şi se obţine $I = \frac{1}{4}\sin 2x + \frac{1}{8}\sin 4x + C$.
2.2. Cu $a = b$ în formulele din exercițiul anterior, rezultă noile formule

$$\cos^2 a = \frac{1+\cos 2a}{2}, \qquad \sin^2 a = \frac{1-\cos 2a}{2}.$$

a) Vom obţine

$$I = \int \cos^2 x\,\mathrm{d}x = \frac{1}{2}\int (1+\cos 2x)\,\mathrm{d}x = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C.$$

b) Rezultă

$$I = \int \sin^2 3x\,\mathrm{d}x = \frac{1}{2}\int (1-\cos 6x)\,\mathrm{d}x = \frac{1}{2}\left(x - \frac{\sin 6x}{6}\right) + C.$$

c) Reducem gradul succesiv.

$$I = \int \sin^4 x\,\mathrm{d}x = \int \left(\frac{1-\cos 2x}{2}\right)^2 \mathrm{d}x = \int \frac{1 - 2\cos 2x + \cos^2 2x}{4}\,\mathrm{d}x$$
$$= \frac{1}{4}\left(x - \sin 2x + \int \cos^2 2x\,\mathrm{d}x\right) = \frac{1}{4}\left[x - \sin 2x + \frac{1}{2}\left(x + \frac{\sin 4x}{4}\right)\right] + C.$$

d) Pentru a integra funcţii trigonometrice la puteri impare procedăm în felul următor.

$$I = \int \cos^5 x\,\mathrm{d}x = \int \cos^4 x \cdot \cos x\,\mathrm{d}x = \int (1-\sin^2 x)^2 \cdot \cos x\,\mathrm{d}x.$$

Cu schimbarea de variabilă $u = \sin x$, avem $\mathrm{d}u = \cos x \, \mathrm{d}x$ şi

$$I = \int (1-u^2)^2 \, \mathrm{d}u = \int (1 - 2u^2 + u^4) \, \mathrm{d}u = u - \frac{2u^3}{3} + \frac{u^5}{5} = \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + C.$$

2.3. Pentru integrale de forma $\int R(\sin x, \cos x) \, \mathrm{d}x$, dacă se verifică condiţia

$$R(-\sin x, \cos x) = -R(\sin x, \cos x)$$

atunci se face schimbarea de variabilă $u = \cos x$. Toate integralele din acest exerciţiu verifică această condiţie. a) Avem

$$I = \int \frac{\sin^3 x}{2 + \cos x} \, \mathrm{d}x = \int \frac{(1-u^2)(-\,\mathrm{d}u)}{2+u} = \int \frac{u^2 - 1}{u+2} \, \mathrm{d}u = \int \frac{u^2 - 4 + 3}{u+2} \, \mathrm{d}u$$
$$= \int \left(u - 2 + \frac{3}{u+2} \right) \mathrm{d}u = \frac{u^2}{2} - 2u + 3\ln|u+2| = \frac{\cos^2 x}{2} - 2\cos x + 3\ln(2 + \cos x) + C.$$

b) Cu schimbarea de variabilă $u = \cos x$ avem

$$I = \int \frac{\sin x}{1 + \cos x + \cos^2 x} = -\int \frac{\mathrm{d}u}{u^2 + u + 1} = -\int \frac{\mathrm{d}u}{\left(u + \frac{1}{2}\right)^2 + \frac{3}{4}}$$
$$= -\frac{2}{\sqrt{3}} \operatorname{arctg} \frac{u + \frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} \operatorname{arctg} \frac{1 + 2\cos x}{\sqrt{3}} + C.$$

c) Aplicând formula $\int \frac{\mathrm{d}x}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right|$

$$I = \int \frac{\sin x}{\cos 2x} = -\int \frac{\mathrm{d}u}{2u^2 - 1} = -\frac{1}{2} \int \frac{\mathrm{d}u}{u^2 - \frac{1}{2}} = -\frac{\sqrt{2}}{4} \ln \left| \frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1} \right| + C.$$

d) După schimbarea de variabilă $u = \cos x$ rezultă

$$I = \int \frac{\mathrm{d}x}{\sin x + \sin 2x} = \int \frac{\mathrm{d}x}{\sin x(1 + 2\cos x)} = -\int \frac{\mathrm{d}u}{(1-u^2)(1+2u)}.$$

Acum se descompune în fracţii simple

$$\frac{-1}{(1-u^2)(1+2u)} = \frac{A}{1-u} + \frac{B}{1+u} + \frac{C}{1+2u},$$

cu $A = -1/6$, $B = 1/2$ şi $C = -4/3$. Rezultă

$$I = \frac{1}{6}\ln(1 - \cos x) + \frac{1}{2}\ln(1 + \cos x) - \frac{2}{3}\ln(1 + 2\cos x) + C.$$

2.4. Când are loc condiţia $R(\sin x, -\cos x) = -R(\sin x, \cos x)$ atunci notăm $u = \sin x$. Toate integralele din acest exerciţiu verifică această condiţie. a) Avem

$$I = \int \frac{\cos x}{\sin x} \, \mathrm{d}x = \int \frac{\mathrm{d}u}{u} = \ln|u| = \ln|\sin x| + C.$$

b)

$$I = \int \frac{\cos^3 x}{\sin^4 x} \, \mathrm{d}x = \int \frac{1 - u^2}{u^4} \, \mathrm{d}u$$
$$= \int u^{-4} \, \mathrm{d}u - \int u^{-2} \, \mathrm{d}u = \frac{u^{-3}}{-3} - \frac{u^{-1}}{-1} = \frac{-1}{3\sin^3 x} + \frac{1}{\sin x} + C.$$

c)

$$I=\int\frac{\mathrm{d}x}{\cos x\cos 2x}=\int\frac{\mathrm{d}u}{(1-u^2)(1-2u^2)}=\int\frac{\mathrm{d}u}{u^2-1}-\int\frac{2\,\mathrm{d}u}{2u^2-1}$$
$$=\frac{1}{2}\ln\frac{1-\sin x}{1+\sin x}-\frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}\sin x-1}{\sqrt{2}\sin x+1}\right|+C.$$

d)

$$I=\int\frac{\sin 2x\,\mathrm{d}x}{(2+\sin x)^2}=\int\frac{2u\,\mathrm{d}u}{(2+u)^2}=2\int\frac{u+2-2\,\mathrm{d}u}{(2+u)^2}$$
$$=2\ln(2+\sin x)+\frac{4}{2+\sin x}+C.$$

2.5. Atunci când este îndeplinită condiţia $R(-\sin x,-\cos x)=R(\sin x,\cos x)$ se face substituţia $u=\operatorname{tg} x$ şi se ţine cont că $\mathrm{d}x=\frac{\mathrm{d}u}{1+u^2}$, $\sin x=\frac{u}{\sqrt{1+u^2}}$ şi $\cos x=\frac{1}{\sqrt{1+u^2}}$.
a)

$$I=\int\frac{\sin^2 x}{\cos^6 x}\,\mathrm{d}x=\int\frac{\frac{u^2}{1+u^2}}{\frac{1}{(1+u^2)^3}}\frac{\mathrm{d}u}{1+u^2}=\int u^2(1+u^2)\,\mathrm{d}u=\int(u^2+u^4)\,\mathrm{d}u=\frac{\operatorname{tg}^3 x}{3}+\frac{\operatorname{tg}^5 x}{5}+C.$$

b)
$$I=\int\frac{\mathrm{d}x}{(\sin x+\cos x)^2}=\int\frac{\mathrm{d}u}{(u+1)^2}=-\frac{1}{u+1}=-\frac{1}{1+\operatorname{tg} x}+C.$$
c)
$$I=\int\frac{\mathrm{d}x}{\cos^4 x+\sin^4 x}=\int\frac{(1+u^2)\,\mathrm{d}u}{1+u^4}=\int\frac{\left(1+\frac{1}{u^2}\right)\mathrm{d}u}{u^2+\frac{1}{u^2}}.$$
Cu schimbarea de variabilă $t=u-\frac{1}{u}$ avem $\mathrm{d}t=\left(1+\frac{1}{u^2}\right)\mathrm{d}u$ şi $t^2=u^2+\frac{1}{u^2}-2$. Aşadar

$$I=\int\frac{\mathrm{d}t}{t^2+2}=\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{t}{\sqrt{2}}=\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{u-\frac{1}{u}}{\sqrt{2}}=\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{\operatorname{tg} x-\operatorname{ctg} x}{\sqrt{2}}+C.$$

d)

$$I=\int\frac{\mathrm{d}x}{\sin^2 x\cos^4 x}=\int\frac{(1+u^2)^2\,\mathrm{d}u}{u^2}=-\frac{1}{u}+2u+\frac{u^3}{3}=-\operatorname{ctg} x+2\operatorname{tg} x+\frac{\operatorname{tg}^3 x}{3}+C.$$

2.6. În cazul în care nici o condiţie de la exerciţiile anterioare nu este verificată se face substituţia $u=\operatorname{tg}\frac{x}{2}$ şi se ţine cont de faptul că $\mathrm{d}x=\frac{2\,\mathrm{d}u}{1+u^2}$, $\sin x=\frac{2u}{1+u^2}$ şi $\cos x=\frac{1-u^2}{1+u^2}$.
a) Avem

$$I=\int\frac{\mathrm{d}x}{5+4\sin x}=2\int\frac{\mathrm{d}u}{5+5u^2+8u}=\frac{2}{5}\int\frac{\mathrm{d}u}{\left(u+\frac{4}{5}\right)^2+\frac{9}{25}}=\frac{2}{3}\operatorname{arctg}\frac{4+5\operatorname{tg}\frac{x}{2}}{3}+C.$$

b)

$$I=\int\frac{\mathrm{d}x}{5-3\cos x}=\int\frac{2\,\mathrm{d}u}{5(1+u^2)-3(1-u^2)}=\int\frac{\mathrm{d}u}{1+4u^2}=\frac{\operatorname{arctg}2u}{2}=\frac{\operatorname{arctg}\left(2\operatorname{tg}\frac{x}{2}\right)}{2}+C.$$