

Seminar 3

Integrale improprii

Să se studieze convergența integralelor și în caz de convergență să se determine valoarea acestora

$$3.1. \int_0^{\infty} \frac{x}{(x+1)(x+2)} dx$$

$$3.2. \int_0^{\infty} \frac{x^{\sqrt{2}} dx}{x^2 + 1}.$$

$$3.3. \int_0^{\infty} \frac{x}{(x+1)(x+2)^2} dx$$

$$3.4. \int_0^1 \frac{x(\arcsin x)^2}{\sqrt{1-x^2}} dx.$$

$$3.5. \int_0^{\frac{\pi}{2}} \ln(\sin x) dx.$$

$$3.6. \int_0^1 \frac{\ln x}{\sqrt{1-x^2}} dx.$$

$$3.7. \int_0^{\infty} \frac{\ln x}{x^2 + 1} dx.$$

$$3.8. \int_0^{\infty} \frac{\operatorname{arctg} x}{x^2 + x + 1} dx.$$

Indicații la problemele propuse

3.1. $\alpha = 1$ DIV

3.2. $\alpha = 2 - \sqrt{2} < 1$ DIV

3.3. $\alpha = 2 > 1$ CONV. Pentru calcul se descompune în fracții simple

$$\frac{x}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

cu $A = -1$, $B = 1$, $C = 2$. Avem

$$\begin{aligned} \int_0^{\infty} \frac{x}{(x+1)(x+2)^2} dx &= \lim_{t \rightarrow \infty} \int_0^t \frac{x}{(x+1)(x+2)^2} dx \\ &= \lim_{t \rightarrow \infty} \int_0^t \frac{-1}{x+1} dx + \int_0^t \frac{1}{x+2} dx + \int_0^t \frac{2}{(x+2)^2} dx \\ &= \lim_{t \rightarrow \infty} \ln \frac{t+2}{t+1} - \ln 2 - \frac{2}{t+2} + 1 = 1 - \ln 2. \end{aligned}$$

3.4. $\alpha = \frac{1}{2}$ CONV. Pentru calcul facem substituția $x = \sin t$. Se obține

$$\int_0^1 \frac{x(\arcsin x)^2}{\sqrt{1-x^2}} dx = \int_0^{\frac{\pi}{2}} t^2 \sin t dt = \pi - 2.$$

3.5. $\alpha = 1/2$ CONV Pentru a o calcula facem schimbarea de variabilă $t = \frac{\pi}{2} - x$ și rezultă

$$I = \int_0^{\frac{\pi}{2}} \ln(\sin x) dx = \int_0^{\frac{\pi}{2}} \ln(\cos x) dx.$$

Atunci

$$\begin{aligned} 2I &= \int_0^{\frac{\pi}{2}} \ln(\sin x) dx + \int_0^{\frac{\pi}{2}} \ln(\cos x) dx = \int_0^{\frac{\pi}{2}} \ln(\sin x \cos x) dx = \int_0^{\frac{\pi}{2}} \ln\left(\frac{\sin 2x}{2}\right) dx \\ &= \int_0^{\frac{\pi}{2}} \ln(\sin 2x) dx - \int_0^{\frac{\pi}{2}} \ln 2 dx \stackrel{2x=t}{=} \frac{1}{2} \int_0^{\pi} \ln \sin t dt - \frac{\pi}{2} \ln 2 \\ &= \frac{1}{2} \left(\int_0^{\frac{\pi}{2}} \ln \sin t dt + \int_{\frac{\pi}{2}}^{\pi} \ln \sin t dt \right) - \frac{\pi}{2} \ln 2 \stackrel{u=\pi-t}{=} \frac{1}{2}(I + I) - \frac{\pi}{2} \ln 2. \end{aligned}$$

Rezultă $I = -\frac{\pi}{2} \ln 2$.

3.6. $\alpha = 1/2$ CONV Cu substituția $x = \sin t$, obținem

$$\int_0^1 \frac{\ln x}{\sqrt{1-x^2}} dx = \int_0^{\frac{\pi}{2}} \ln(\sin t) dt = -\frac{\pi}{2} \ln 2.$$

3.7. $\alpha = 3/2$ CONV Cu substituția $x = 1/t$ obținem $I = -I$, adică $I = 0$.

3.8. $\alpha = 2$ CONV Cu substituția $x = 1/t$ rezultă

$$I = \int_0^{\infty} \frac{\operatorname{arctg} \frac{1}{t}}{t^2 + t + 1} dt.$$

Folosind faptul că $\operatorname{arctg} t + \operatorname{arctg} \frac{1}{t} = \frac{\pi}{2}$, pentru orice $t > 0$,

$$2I = \int_0^{\infty} \frac{\operatorname{arctg} t}{t^2 + t + 1} dt + \int_0^{\infty} \frac{\operatorname{arctg} \frac{1}{t}}{t^2 + t + 1} dt = \frac{\pi}{2} \int_0^{\infty} \frac{1}{t^2 + t + 1} dt = \frac{\pi^2}{3\sqrt{3}}.$$

Rezultă $I = \frac{\pi^2}{6\sqrt{3}}$.

Probleme propuse

Să se studieze convergența integralelor:

Problema 3.9. $\int_1^{\infty} \frac{\sqrt{x+1}}{x^3 \sqrt{x^2+1}} dx.$

Problema 3.10. $\int_0^{\infty} \frac{\operatorname{arctg} x}{x\sqrt{x}} dx.$

Problema 3.11. $\int_1^{\infty} \frac{x}{(x-2)(x^2+1)} dx.$

Problema 3.12. $\int_1^e \frac{1}{(x+1)\sqrt{\ln x}} dx.$

Problema 3.13. $\int_0^1 \frac{e^{-\frac{1}{x}}}{x^3} dx.$

Problema 3.14. $\int_1^\infty \frac{(\ln x)^4}{(x-1)^3} dx.$

Problema 3.15. $\int_0^\infty \frac{\sin x}{x\sqrt{x}} dx.$

Problema 3.16. $\int_2^\infty \frac{\cos x}{(x-1)^2} dx.$

Problema 3.17. $\int_a^b \frac{1}{\sqrt{x-a}} \cdot \sin \frac{1}{(x-a)^2} dx.$

Problema 3.18. $\int_a^b \frac{x^3}{\sqrt{(x-a)(b-x)}} dx.$

Să se calculeze integralele:

Problema 3.19. $\int_0^\infty \frac{x^2}{(x+1)(x+2)^2(x+3)} dx$

Problema 3.20. $\int_0^\infty \frac{dx}{(x^2+4)^n}.$

Problema 3.21. $\int_{-\infty}^\infty \frac{x^2+1}{(x^2+4)(x^2+9)} dx.$

Problema 3.22. $\int_0^\infty \frac{x+1}{(x+2)^2(x+3)} dx.$

Problema 3.23. $\int_0^\infty \frac{dx}{(x^2+1) \cdots (x^2+n)}.$

Problema 3.24. $\int_a^b \frac{\sqrt{x-a}}{\sqrt{b-x}} dx.$

Problema 3.25. $\int_a^b \frac{x}{\sqrt{(x-a)(b-x)}} dx.$

Problema 3.26. $\int_a^b \frac{x^2}{\sqrt{(x-a)(b-x)}} dx.$

Problema 3.27. $\int_{-1}^1 \frac{x}{\sqrt{1-x^2}} dx.$

Problema 3.28. $\int_0^1 \frac{1}{\sqrt{x-x^2}} dx.$

Problema 3.29. $\int_0^\infty \frac{\ln x}{x^2+x+1} dx.$

Problema 3.30. $\int_0^\infty \frac{\operatorname{arctg} x}{x^2+1} dx.$

Problema 3.31. $\int_1^\infty \frac{\ln(1+x)}{x^3} dx.$

Problema 3.32. $\int_0^\infty x^{2n+1} e^{-x^2} dx.$