Matematici speciale ETTI restanțe toamnă 2024

Subiecte I

1. Funcția Gamma (definiție + 3 proprietăți)

2. $\displaystyle\int_1^{\infty} \frac{1}{(x^2+1)(x^2+3)}\,dx$ convergență + calcul

3. $\displaystyle\int_{|z|=3} \frac{e^{-\pi z}}{(z-1)^2\cdot(z^2+4)}\,dz$

4. $\displaystyle\iint_D y\cdot\sqrt{x^2+y^2}\,dx\,dy$, $D: x^2+y^2\le 3$, $y\geq 0$.

Posibilă rezolvare a subiectelor

1. Funcția Gamma $\Gamma : (0, \infty) \to \mathbb{R}$

$$\Gamma(a) = \int_0^\infty e^{-x} \cdot x^{a-1} \, dx \quad , \quad a > 0 .$$

Proprietăți:

1) $\Gamma(1) = 1$

2) $\Gamma(n) = (n-1)! \quad , \quad n \in \mathbb{N}, \ n \geq 1$

3) $\Gamma(a) = (a-1)\Gamma(a-1) \quad , \quad a > 1$

4) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

5) $\Gamma(a) \cdot \Gamma(1-a) = \dfrac{\pi}{\sin(\pi a)} \quad , \quad a \in (0,1).$

2. **convergența integralei** $\int_1^\infty \frac{1}{(x^2+1)(x^2+3)}\,dx$

$$L = \lim_{x\to\infty} x^\alpha \cdot \frac{1}{(x^2+1)(x^2+3)} \underset{\alpha=4}{=} \lim_{x\to\infty} \frac{\cancel{x^4}}{\cancel{x^2\left(1+\frac{1}{x^2}\right)x^2}\left(1+\frac{3}{x^2}\right)} = 1$$

Pt că $\alpha = 4 > 1$ și $L = 1 > 0$ $\Rightarrow$ integrala este convergentă

Calculul integralei

Descompunem în fracții simple

$$\frac{1}{(x^2+1)(x^2+3)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+3}$$

Pt că în membrul drept variabila x apare doar în forma x^2

putem scrie mai pe scurt

$$\frac{1}{(x^2+1)(x^2+3)} = \frac{A}{x^2+1} + \frac{B}{x^2+3} \quad \Big| \cdot (x^2+1)(x^2+3)$$

$$\Rightarrow \quad 1 = A(x^2+3) + B(x^2+1)$$

$$x^2 = -1 \Rightarrow 1 = A\cdot(-1+3) \Rightarrow A = \frac{1}{2}$$

$$x^2 = -3 \Rightarrow 1 = B\cdot(-3+1) \Rightarrow B = -\frac{1}{2}.$$

$$\Rightarrow \int_1^\infty \frac{1}{(x^2+1)(x^2+3)}\,dx = \frac{1}{2}\int_1^\infty \frac{1}{1+x^2}\,dx - \frac{1}{2}\int_1^\infty \frac{1}{x^2+3}\,dx$$

$$= \frac{1}{2}\operatorname{arctg} x\Big|_1^\infty - \frac{1}{2\sqrt{3}}\operatorname{arctg}\frac{x}{\sqrt{3}}\Big|_1^\infty = \frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) - \frac{1}{2\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{6}\right) = \frac{\pi}{8} - \frac{\pi}{6\sqrt{3}}.$$

3. $(z-1)^2 \cdot (z^2+4) = 0 \Rightarrow z_1 = 1$ pol de ordinul $k=2.$

$z^2+4=0 \quad z_2 = 2i, \; z_3 = -2i$ poli de ordin $k=1$

$|z_1| = |1| = 1 < 3 \Rightarrow z_1 \in$ interiorului cercului $|z|=3$

$|z_{2,3}| = |\pm 2i| = 2 < 3 \Rightarrow z_{2,3} \in$ interiorul cercului $|z|=3.$

$$\Rightarrow \int_{|z|=3} \frac{e^{-\pi z}}{(z-1)^2(z^2+4)}\,dz = 2\pi i \cdot \operatorname{Rez}(f, z_1) + 2\pi i \operatorname{Rez}(f, z_2) + 2\pi i \cdot \operatorname{Rez}(f, z_3)$$

Formula pentru calculul reziduului unui pol de ordin k este

$$\operatorname{Rez}(f, a) = \lim_{z \to a} \frac{1}{(k-1)!}\left[(z-a)^k \cdot f(z)\right]^{(k-1)}$$

$$\operatorname{Rez}(f, z_1) = \lim_{z \to 1} \left(\cancel{(z-1)^2} \cdot \frac{e^{-\pi z}}{\cancel{(z-1)^2}(z^2+4)}\right)' = \lim_{z \to 1} \frac{-\pi e^{-\pi z} \cdot (z^2+4) - e^{-\pi z} \cdot 2z}{(z^2+4)^2} = \frac{-e^{-\pi}(5\pi+2)}{25}$$

$$\operatorname{Rez}(f, z_2) = \lim_{z \to 2i} \frac{(z - z_2) \cdot e^{-\pi z}}{(z-1)^2 \cdot z^2 + 4} = \lim_{z \to 2i} \frac{\cancel{(z-2i)}\, e^{-\pi z}}{(z-1)^2 \cancel{(z-2i)}(z+2i)}$$

$$= \frac{e^{-2\pi i}}{(2i-1)^2 \cdot 4i} = \frac{1}{(2i-1)^2 4i}$$

pt că $e^{-2\pi i} = \cos(-2\pi) + i\sin(-2\pi)$
$= 1$

$$\operatorname{Rez}(f, z_3) = \lim_{z \to -2i} \frac{(z - z_3)\, e^{-\pi z}}{(z-1)^2 (z^2+4)} = \frac{e^{2\pi i}}{(-2i-1)^2(-2i-2i)} = -\frac{1}{4i(1+2i)^2}$$

$$\Rightarrow I = -\frac{2\pi i\, e^{-\pi}(5\pi+2)}{25} + \frac{2\pi i}{4i(2i-1)^2} - \frac{2\pi i}{4i(1+2i)^2}$$

$$= -\frac{2\pi i \cdot e^{-\pi}(5\pi+2)}{25} + \frac{\pi}{2}\left(\frac{1}{(2i-1)^2} - \frac{1}{(2i+1)^2}\right)$$

$$= -\frac{2\pi i\, e^{-\pi}(5\pi+2)}{25} + \frac{4\pi i}{25}$$

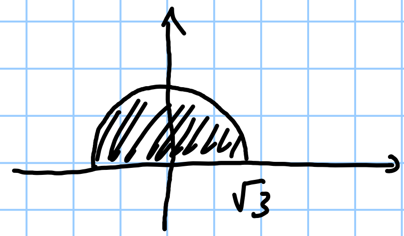
$$\frac{(2i+1)^2 - (2i-1)^2}{(2i-1)^2 \cdot (2i+1)^2}$$

$$= \frac{-4+4i+1+4+4i-1}{[(2i-1)(2i+1)]^2}$$

$$= \frac{8i}{(-4-1)^2} = \frac{8i}{25}$$

4. $x^2+y^2 \le 3$ interiorul unui cerc centrat în O și de rază $\sqrt{3}$.

$y \geq 0$ semiplanul deasupra axei Ox.



Folosim coordonate polare

$$x = \rho\cos\varphi \qquad \rho \in [0, \sqrt{3}]$$

$$y = \rho\sin\varphi \qquad \varphi \in [0, \pi]$$

$$|J| = \rho$$

$$\Rightarrow \quad I = \iint_D y\sqrt{x^2+y^2}\,dx\,dy = \int_0^{\sqrt{3}}\int_0^{\pi} \rho\sin\varphi\cdot\sqrt{\rho^2}\cdot\rho\,d\rho\,d\varphi$$

$$= \int_0^{\sqrt{3}} \rho^3\,d\rho \cdot \int_0^{\pi}\sin\varphi\,d\varphi = \left.\frac{\rho^4}{4}\right|_0^{\sqrt{3}} \cdot \left.(-\cos\varphi)\right|_0^{\pi} =$$

$$= \frac{(\sqrt{3})^4}{4}\cdot(-\cos\pi+\cos 0) = \frac{3^2}{4}\cdot(1+1) = \frac{9}{4}\cdot 2 = \frac{9}{2}.$$