Seminar 1

Integrale duble

Să se calculeze

Problema 1.1. $\iint_D xy^2 \,\mathrm{d}x\,\mathrm{d}y$, unde D este interiorul triunghiului ABC, $A(0,0)$, $B(1,0)$ şi $C(0,1)$.

Problema 1.2. $\iint_D 4x \,\mathrm{d}x\,\mathrm{d}y$, unde D este interiorul triunghiului ABC, $A(1,1)$, $B(3,3)$ şi $C(1,4)$.

Problema 1.3. $\iint_D \frac{\ln(x^2+y^2)}{x^2+y^2} \,\mathrm{d}x\,\mathrm{d}y$, unde D este $1 \leq x^2+y^2 \leq e^2$.

Problema 1.4. $\iint_D x \,\mathrm{d}x\,\mathrm{d}y$, unde $D = \{\, (x,y) \mid x^2+y^2 \leq 2,\ y \geq x \,\}$.

Problema 1.5. Aria domeniului mărginit de elipsa $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Indicaţii

1.1.

$$I = \int_0^1 \left(\int_0^{1-x} xy^2 \,\mathrm{d}y \right) = \frac{1}{3} \int_0^1 x(1-x)^3 \,\mathrm{d}x = \frac{1}{3}\beta(2,4) = \frac{1}{3}\frac{\Gamma(2)\Gamma(4)}{\Gamma(6)} = \frac{1}{60}.$$

1.2.

$$I = \int_1^3 \left(\int_x^{\frac{9-x}{2}} 4x \,\mathrm{d}y \right) \mathrm{d}x = \int_1^3 4x \left(\frac{9-x}{2} - x \right) \mathrm{d}x = \int_1^3 (18x - 6x^2) \,\mathrm{d}x = 20.$$

1.3. Se face schimbarea de variabile la coordonate polare $x = \rho\cos\varphi$, $y = \rho\sin\varphi$. Noul domeniu este $(\rho,\varphi) \in \Delta = [1,e] \times [0,2\pi]$, iar $|J| = \rho$. Valoarea integralei este

$$\iint_\Delta \frac{\ln\rho^2}{\rho^2} \cdot |J| \,\mathrm{d}\rho\,\mathrm{d}\varphi = \left(\int_1^e \frac{2\ln\rho}{\rho} \,\mathrm{d}\rho \right) \cdot \left(\int_0^{2\pi} \mathrm{d}\varphi \right) = 2\pi \ln^2 \rho \Big|_1^e = 2\pi.$$

1.4. Se face schimbarea de variabile la coordonate polare $x = \rho\cos\varphi$, $y = \rho\sin\varphi$. Noul domeniu este $(\rho,\varphi) \in \Delta = [0,\sqrt{2}] \times [\frac{\pi}{4}, \pi + \frac{\pi}{4}]$, iar $|J| = \rho$. Integrala se calculează

$$\iint_D x \,\mathrm{d}x\,\mathrm{d}y = \iint_\Delta \rho\cos\varphi |J| \,\mathrm{d}\rho\,\mathrm{d}\varphi = \left(\int_0^{\sqrt{2}} \rho^2 \,\mathrm{d}\rho \right) \cdot \left(\int_{\frac{\pi}{4}}^{\pi+\frac{\pi}{4}} \cos\varphi \,\mathrm{d}\varphi \right) = \frac{2\sqrt{2}}{3} \cdot (-\sqrt{2}) = -\frac{4}{3}.$$

1.5. Se face schimbarea de variabile la coordonate polare generalizate $x = a\rho\cos\varphi$, $y = b\rho\sin\varphi$. Noul domeniu este $(\rho, \varphi) \in \Delta = [0, 1] \times [0, 2\pi]$, iar $|J| = ab\rho$. Aria va fi

$$Aria(\text{elipsei}) = \iint_D \mathrm{d}x\,\mathrm{d}y = \iint_\Delta |J|\,\mathrm{d}\rho\,\mathrm{d}\varphi = ab\left(\int_0^1 \rho\,\mathrm{d}\rho\right)\cdot\left(\int_0^{2\pi}\mathrm{d}\varphi\right) = \pi ab.$$