Let $A \subseteq \mathbb{R}$. Consider a functional sequence $(f_n)$ of real functions defined on $A$ and $f : A \to \mathbb{R}$.

Define the Chebyshev norm of $f$, $\|f\| := \sup_{x \in A} |f(x)|$.

**D 1** A functional sequence $(f_n)$ is **convergent** to $f$ on $A$, and denoted by $f_n \longrightarrow f$, if

**D 2** A functional sequence $(f_n)$ is said to be **uniformly convergent** to $f$ on $A$, denoted by $f_n \overset{U}{\longrightarrow} f$, if
Uniform convergence

If the functions \( f_n : [a, b] \to \mathbb{R} \) are continuous and if \( f_n \xrightarrow{U} f \), then \( f \) is continuous on the closed interval \([a, b]\).

\[
f_n(x) = \begin{cases} 
  e^n & \text{if } x \in [0, 1) \\
  1 & \text{if } x = 1 
\end{cases}
\]

is an example of non-uniform convergence.
Let \((f_n)\) be a uniformly convergent sequence of continuous functions on the interval \([a, b]\). Then the following equality holds

\[
\lim_{n \to \infty} \int_{a}^{b} f_n(x) \, dx \equiv \int_{a}^{b} \lim_{n \to \infty} f_n(x) \, dx.
\]

By virtue of the previous theorem we deduce:

A uniformly convergent series of continuous functions on a closed interval \([a, b]\) can be integrated \(\text{term-by-term}\).

Example: \(f_n : [0, 1] \to \mathbb{R},\)

\[
f_n(x) = (n + 1)(n + 2) \,(x^n - x^{n+1}).
\]

\[
\lim_{n \to \infty} f_n(x) = 0, \quad x \in [0, 1].
\]

\[
\int_{0}^{1} f_n(x) \, dx = \int_{0}^{1} (n + 1)(n + 2) \,(x^n - x^{n+1}) \, dx = 1
\]

\[
\lim_{n \to \infty} \int_{0}^{1} f_n(x) \, dx \neq \int_{0}^{1} \lim_{n \to \infty} f_n(x) \, dx = \int_{0}^{1} 0 \, dx = 0
\]

It follows that \((f_n)\) does not converge uniformly.
Let \((f_n)\) be a sequence of functions in \(C^1[a, b]\). If \((f_n)\) converges to a function \(f\), and the sequence of the derivatives \((f'_n)\) is uniformly convergent to a function \(g\), then there exists \(f'\) and \(f' = g\), i.e.,

\[
\lim_{n \to \infty} (f_n)' = (\lim_{n \to \infty} f_n)'.
\]

(Weierstrass) If \(\sum a_n\) is a convergent series of numbers, and the functional sequence of real functions \((f_n)\), satisfies the inequalities

\[
|f_n(x)| \leq a_n, \quad \forall x \in A, \forall n \in \mathbb{N},
\]

then the functional series \(\sum f_n\) is

uniformly and absolutely convergent on \(A\).

**Power Series**

Let \((a_n)\) be a sequence of complex numbers and \(z_0 \in \mathbb{C}\).

The series

\[
\sum_{n \geq 0} a_n (z - z_0)^n
\]

is called

Making the substitution \(x = z - z_0\) we obtain a power series in power of \(x\), \(\sum_{n \geq 0} a_n x^n\). Some features of such series will be given.
(Abel) If a power series $\sum a_n x^n$ converges at a point $x_0 \neq 0$, then it converges uniformly and absolutely in every closed disc $D(0, r) := \{ x \in \mathbb{C} \mid |x| \leq r \}$, $0 < r < |x_0|$.

Let $x \in D$. Since the series $\sum a_n x_0^n$ is convergent, there exists a number $M$ such that $|a_n x_0^n| \leq M$, $\forall n \in \mathbb{N}$.

We have:

$$\sum |a_n x^n| = \sum |a_n x_0^n| \left| \frac{x}{x_0} \right|^n \leq M \sum \left| \frac{r}{x_0} \right|^n, \quad \forall x \in D.$$ 

for all $x \in D$. By Weierstrass's Theorem, the series $\sum a_n x^n$ is uniformly and absolutely convergent in $D$.

We use recall the notation

$$\lim a_n = \sup \{ \text{the limit points set of } (a_n) \}$$

The radius of convergence of a power series is the radius of the largest disk in the interior of which the series converges. It is either a non-negative real number or $\infty$.

We define $R := \frac{1}{\lim \sqrt[n]{|a_n|}} \in [0, \infty]$, where

The next theorem states that the element $R$ in D11 is the radius of convergence of the series $\sum a_n x^n$. 

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• If $|x| < R$, then $\sum |a_n x^n|$ is convergent;

• If $|x| > R$, then $\sum a_n x^n$ is divergent.

We use the Cauchy Root Test. The set

$$D(0, R) = \left\{ x \in \mathbb{C} \mid |x| < R \right\}$$

is the disc of convergence of the series.

---

Find the radius of convergence of the power series

$$\sum_{n \geq 0} x^n n!$$

We have:

$$R = \frac{1}{\lim_{n \to \infty} \sqrt[n]{\frac{1}{n!}}} = \lim_{n \to \infty} \sqrt[n]{n!} = \lim_{n \to \infty} \frac{(n+1)!}{n!}$$

$$= \lim_{n \to \infty} (n + 1) = \infty$$

✓
Find the radius of convergence of the power series

\[ \sum_{n \geq 0} n! x^n. \]

We have:

\[
R = \frac{1}{\lim_{n \to \infty} \sqrt[n]{n!}} = \lim_{n \to \infty} \frac{n!}{(n + 1)!} = \lim_{n \to \infty} \frac{1}{n + 1} = 0.
\]

The series is convergent only at the point \( x = 0 \).

When the limit \( \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| \) exists, then the radius of convergence \( R \) of the series \( \sum a_n x^n \) can be computed by

\[
R = \frac{1}{\lim_{n \to \infty} \sqrt[n]{|a_n|}}.
\]

(Abel) If \( 0 < R < \infty \) is the radius of convergence of a series \( \sum a_n x^n \) and if the number series \( \sum a_n R^n \) is convergent, then
Since the series
\[ \sum a_n x^n, \quad \sum n a_n x^{n-1}, \quad \text{and} \quad \sum \frac{a_n}{n+1} x^{n+1}, \]
have the same radius of convergence, any power series on the domain of convergence.

Functions of complex variable which can be expanded into convergent power series are called analytic functions. They are studied as a branch of advanced mathematics called the Theory of Analytic Functions or The Theory of Functions of Complex Variables.

Some of these functions will be presented.

\[ e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \]

\[ \cos z := \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \]

\[ \sin z := \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \]

For real \( z = x \) these definitions lead to the well-known real functions \( e^x, \cos x \) and \( \sin x, \quad x \in \mathbb{R} \).
It is obvious that for any complex number $z$ we have

$$e^{iz} = \cos z + i \sin z,$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

(Euler’s formulas). ✓

---

**Trigonometric Series. Fourier Series**

\[
\begin{align*}
\sin a \cos b &= \frac{1}{2}(\sin(a - b) + \sin(a + b)) \\
\cos a \cos b &= \frac{1}{2}(\cos(a - b) + \cos(a + b)) \\
\sin a \sin b &= \frac{1}{2}(\cos(a - b) - \cos(a + b))
\end{align*}
\]
\[
\int_{-\pi}^{\pi} \cos(kx) \sin(nx) \, dx = 0, \quad k, n = 0, 1, \ldots
\]

\[
\int_{-\pi}^{\pi} \cos(kx) \cos(nx) \, dx = 0,
\quad k \neq n, \quad k + n \neq 0
\]

\[
\int_{-\pi}^{\pi} \sin(kx) \sin(nx) \, dx = 0,
\quad k \neq n, \quad k + n \neq 0
\]

\[
\int_{-\pi}^{\pi} \cos^2(nx) \, dx = \begin{cases} 
\pi, & n = 1, 2, \ldots \\
2\pi, & n = 0
\end{cases}
\]

\[
\int_{-\pi}^{\pi} \sin^2(nx) \, dx = \pi, \quad n = 1, 2, \ldots
\]

Maclaurin

\[
f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + a_{n+1} x^{n+1} + \cdots
\]

\[
f^{(n)}(x) = 0 + 0 + 0 + \cdots + n!a_n + (n + 1)!a_{n+1} x + \cdots \quad x \to 0
\]

\[
f^{(n)}(0) = n!a_n
\]

Fourier

\[
f(x) = a_0 \frac{\sin x}{x} + a_1 \cos x + b_1 \sin x + \cdots + a_n \cos nx + b_n \sin nx + \cdots
\]

\[
\int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0 + a_1 \times 0 + b_1 \times 0 + \cdots + a_n \times 0 + b_n \times \pi + \cdots + 0 + \cdots
\]
Let \((a_n)_{n \geq 0}\) and \((b_n)_{n \geq 1}\) be two sequences of real numbers.

**D 18** The series

\[ a_0^2 + \sum_{n \geq 1} (a_n \cos nX + b_n \sin nX), \]

is called a [ ] of the coefficients \((a_n)\) and \((b_n)\).

Let \(f : \mathbb{R} \to \mathbb{R}\) be a periodic function with period \(2\pi\), Riemann integrable on the interval \([-\pi, \pi]\).

**D 19** The trigonometric series

\[ \frac{a_0}{2} + \sum_{n \geq 1} (a_n \cos nX + b_n \sin nX), \]

where the coefficients \(a_n\) and \(b_n\) are given by

\[ a_n = \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \]

\[ b_n = \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \]

is called the [Fourier series] of the function \(f\), and write

\[ f(x) \sim \ldots \]
(* MATHEMATICA *)

Play[ Sin [440 * 2πt], {t, 0, 1}]

(* MATHEMATICA *)

Sound[
   Play[ Sin [400 * 2Pi(t - 1)^2], {t, 0, 1} ],
   Play[ Sin [300 * 2Pi(t - 1)^2], {t, 0, 1} ],
   Play[ Sin [900 * 2Pi(t - 1)^2], {t, 0, 1} ]
]
If $f$ is an even function, then its Fourier coefficients are given by

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx, \quad n \geq 0,$$

$$b_n = 0, \quad n \geq 1;$$

and if $f$ is an odd function, then its Fourier coefficients are given by

$$a_n = 0, \quad n \geq 0;$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx, \quad n \geq 1.$$

(Dirichlet) If a function $f$ is piecewise smooth, then its Fourier series converges to

for all $x \in \mathbb{R}$. 


Let expand the function $x \mapsto \frac{x}{2}$, $x \in (-\pi, \pi)$, as a Fourier series.

Consider the function $f : \mathbb{R} \to \mathbb{R}$, with period $2\pi$,

$$f(x) = \begin{cases} 
0, & x = -\pi, \\
\frac{x}{2}, & x \in (-\pi, \pi), \\
0, & x = \pi.
\end{cases}$$

Since the function $f$ is odd we get:

$$a_n = 0, \quad n \in \mathbb{N}.$$ 

$$b_n = \frac{2}{\pi} \int_{-\pi}^{\pi} \frac{x}{2} \sin nx \, dx = \frac{(-1)^{n+1}}{n}, \quad n \in \mathbb{N}^*.$$ 

From the equality

$$f(x) = \frac{f(x + 0) + f(x - 0)}{2}, \quad x \in \mathbb{R},$$

using Dirichlet's Theorem 21, we obtain

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, \quad x \in \mathbb{R},$$

hence,

$$\frac{x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, \quad x \in (-\pi, \pi). \quad (1)$$
From Eq (1), for $x = \frac{\pi}{2}$, we deduce

and we re-obtain the Gregory-Leibniz Formula.

Graphs of $S_3$, $S_{10}$, $S_{100}$
If a function \( f \in C^1(\mathbb{R}) \) is periodic, then its Fourier series converges uniformly to \( f \).

The Fourier series associated to any integrable function \( f : [a, b] \rightarrow \mathbb{R} \) is

\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{2n\pi}{b-a}x + b_n \sin \frac{2n\pi}{b-a}x \right),
\]

where the coefficients are given by

\[
a_n = \frac{2}{b-a} \int_{a}^{b} f(x) \cos \frac{2n\pi}{b-a}x \, dx, \quad n \in \mathbb{N},
\]

\[
b_n = \frac{2}{b-a} \int_{a}^{b} f(x) \sin \frac{2n\pi}{b-a}x \, dx, \quad n \in \mathbb{N}^*.
\]
**T 26** (Parseval) If a function $f : [-\pi, \pi] \to \mathbb{R}$ and the square of its modulus are integrable, then its Fourier coefficients satisfy the equality

$$\frac{a_0^2}{2} + \sum_{n \geq 1} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) \, dx$$

**E 27** Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \boxed{\frac{\pi^2}{6}}$$

(cf. ♠1)

$$\frac{x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, \quad x \in (-\pi, \pi),$$

For odd functions, Parseval’s equality becomes:

$$\sum_{n=1}^{\infty} (b_n)^2 = \frac{2}{\pi} \int_{0}^{\pi} (f(x))^2 \, dx$$

hence

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2}{\pi} \int_{0}^{\pi} \frac{x^2}{4} \, dx = \frac{\pi^2}{6}$$

✓
We have (1)

\[
\frac{t}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt, \quad t \in (-\pi, \pi)
\]

With \( t := \pi - x \) we obtain

\[
\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n(\pi - x), \quad x \in (0, 2\pi),
\]

i.e.,

\[
\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx, \quad x \in (0, 2\pi)
\]

\[\boxed{\text{P 28}}\]

\[
\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{\sin nx}{n}, \quad x \in (0, 2\pi).
\]

\[\boxed{\text{P 29}}\]

\[
\cos ax = \frac{2 \sin a\pi}{\pi} \left( \frac{1}{2a} + \sum_{n=1}^{\infty} \frac{(-1)^n a}{a^2 - n^2} \cos nx \right),
\]

\[a \in \mathbb{C} \setminus \mathbb{Z}, \quad x \in [-\pi, \pi].\]

We expand the even function \( x \in [-\pi, \pi], \) with period \( 2\pi \), into a Fourier series.
We have:

\[ b_n = 0, \quad n \in \mathbb{N}^*, \]

\[ a_n = \frac{2}{\pi} \int_0^\pi \cos a x \cos n x \, dx = \frac{1}{\pi} \int_0^\pi (\cos(a + n)x + \cos(a - n)x) \, dx \]

\[ = (-1)^n \frac{2a \sin a\pi}{\pi (a^2 - n^2)}, \quad n \in \mathbb{N}, \]

hence

\[ \cos ax = \frac{2 \sin a\pi}{\pi} \left( \frac{1}{2a} + \sum_{n=1}^{\infty} \frac{(-1)^n a}{a^2 - n^2} \cos nx \right), \quad x \in [-\pi, \pi] \]

\[ \cot a\pi - \frac{1}{a\pi} = \sum_{n=1}^{\infty} \frac{2a\pi}{a^2\pi^2 - n^2\pi^2}. \]
Consequently, for $\alpha = t \in (0, \pi)$, we get

\[
\cot t - \frac{1}{t} = \sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2\pi^2}
\]

Since

\[
\int \frac{2t}{t^2 - n^2\pi^2} \, dt = \log(n^2\pi^2 - t^2) \quad \left( - \log(n^2\pi^2) \right),
\]

we obtain

\[
\log \left( \frac{\sin x}{x} \right) = \log \sin x - \log x = \int_0^x \left( \cot t - \frac{1}{t} \right) \, dt
\]

\[
= \sum_{n=1}^{\infty} \log \left( 1 - \frac{x^2}{n^2\pi^2} \right) = \log \prod_{n=1}^{\infty} \left( 1 - \frac{x^2}{n^2\pi^2} \right)
\]

Resolution of \(1/\sin\) into Partial Fractions

\[
\frac{1}{\sin x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{x - n\pi}, \quad x \in \mathbb{R} \setminus \pi \mathbb{Z}.
\]

Taking \(x = \boxed{\alpha} \) in

\[
\cos ax = \frac{2 \sin a\pi}{\pi} \left( \frac{1}{2a} + \sum_{n=1}^{\infty} \frac{(-1)^n a}{a^2 - n^2} \cos nx \right),
\]

we get

\[
\frac{1}{\sin a\pi} = \frac{1}{a\pi} + \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{a\pi - n\pi} + \frac{1}{a\pi + n\pi} \right)
\]

hence, for \(\boxed{\alpha} = x\), we obtain

\[
\frac{1}{\sin x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{x - n\pi}, \quad x \in \mathbb{R} \setminus \pi \mathbb{Z}
\]
Resolution of $1/\text{Sinc}$ into Partial Fractions

P 31 $\frac{1}{\sin x} = \sum_{n=-\infty}^{\infty} \left( -\frac{1}{x - n\pi} \right)$, $x \in \mathbb{R} \setminus \pi \mathbb{Z}$.

Taking $x = 0$ in

$$\cos ax = \frac{2 \sin a\pi}{\pi} \left( \frac{1}{2a} + \sum_{n=1}^{\infty} \frac{(-1)^n a}{a^2 - n^2} \cos nx \right),$$

we get

$$\frac{1}{\sin a\pi} = \frac{1}{a\pi} + \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{a\pi - n\pi} + \frac{1}{a\pi + n\pi} \right),$$

hence, for $a\pi = x$, we obtain

$$\frac{1}{\sin x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{x - n\pi}, \quad x \in \mathbb{R} \setminus \pi \mathbb{Z}.$$

The Infinite Product Expansion

of the Tangent Function

P 32 $\tan x = x \prod_{n=1}^{\infty} \frac{1 - x^2/n^2\pi^2}{1 - 4x^2/(2n-1)^2\pi^2}$, $x \in (-\pi/2, \pi/2)$.

Taking $x = 0$ in

$$\cos ax = \frac{2 \sin a\pi}{\pi} \left( \frac{1}{2a} + \sum_{n=1}^{\infty} \frac{(-1)^n a}{a^2 - n^2} \cos nx \right),$$

we obtain

$$\frac{1}{\sin a\pi} - \frac{1}{a\pi} = \sum_{n=1}^{\infty} (-1)^n \frac{2\alpha\pi}{(a\pi)^2 - (n\pi)^2}.$$
For $a\pi = 2t \in (0, \pi)$, we get

$$\frac{1}{\sin 2t} - \frac{1}{2t} = \sum_{n=1}^{\infty} (-1)^n \frac{4t}{4t^2 - n^2\pi^2}.$$ 

$$\int \frac{1}{\sin 2t} \, dt = \int \frac{1}{2 \sin t \cos t} \, dt = \frac{1}{2} \int \frac{1}{\cos t \cos^2 t} \, dt = \frac{1}{2} \int \frac{(\tan t)'}{\tan t} \, dt = \frac{1}{2} \log |\tan t|.$$ 

$$\frac{1}{2} \log \frac{\tan x}{x} = \int_0^x \left( \frac{1}{\sin 2t} - \frac{1}{2t} \right) \, dt = \sum_{n=1}^{\infty} (-1)^n \int_0^x \frac{4t}{4t^2 - n^2\pi^2} \, dt = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n \log \left( 1 - \frac{4x^2}{n^2\pi^2} \right) = \frac{1}{2} \log \prod_{n=1}^{\infty} \frac{1 - \frac{x^2}{n^2\pi^2}}{1 - \frac{4x^2}{(2n-1)^2\pi^2}}.$$ 

\[\text{P 33}\]

$$\sum_{n=0}^{\infty} \frac{\cos nx}{n!} = e^{e^{n \pi}} \cos (\sin x),$$

$$\sum_{n=0}^{\infty} \frac{\sin nx}{n!} = e^{e^{n \pi}} \sin (\sin x), \quad x \in \mathbb{R}.$$ 

In $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $z \in \mathbb{C}$, we take $z = e^{ix} = \cos x + i \sin x$, we obtain:

$$e^{e^{ix}} = \sum_{n=0}^{\infty} \frac{(e^{ix})^n}{n!} = \sum_{n=0}^{\infty} \frac{e^{inx}}{n!} = \sum_{n=0}^{\infty} \frac{\cos nx + i \sin nx}{n!},$$

hence

$$e^{e^{ix}}(\cos (\sin x) + i \sin (\sin x)) = \sum_{n=0}^{\infty} \frac{\cos nx}{n!} + i \sum_{n=0}^{\infty} \frac{\sin nx}{n!}.$$