Undirected Graphs

Terminology. Free Trees. Representations. Minimum Spanning Trees (algorithms: Prim, Kruskal). Graph Traversals (dfs, **bfs**). Articulation points & Biconnected Components. Graph Matching

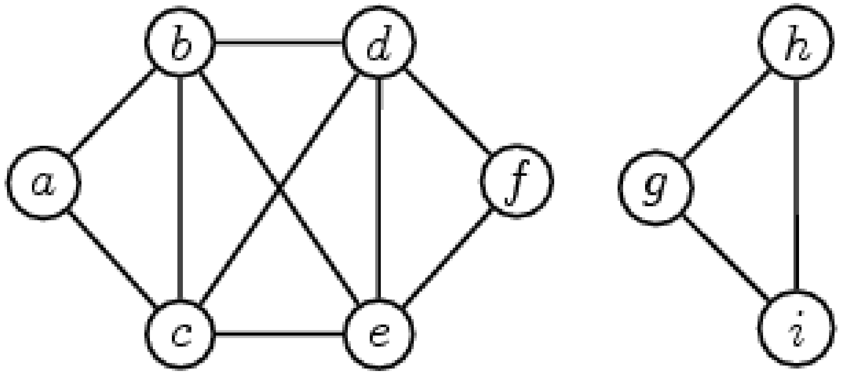
Application domains for graphs

- Graphs provide a useful way to model a large variety of problems in an abstract way
  - Communication networks
  - Computer networks
  - Maps (cities and highways)
  - Path planning in AI (states and moves)
  - Scientific taxonomy
  - Activity charts (tasks and dependencies)
  - Flow chart of computer programs

Undirected graphs

- Undirected graphs (or just graphs): $G=(V, E)$, with V and E *finite* (true for digraphs as well)
  - Differ from digraphs by the fact that each edge is *unordered* (i.e. (u, v) is the same as (v, u))
- Visualizations of graphs (and digraphs) are usually *embeddings*
- An *embedding* maps each node of G to a point in the plane, and each arc to a curve or straight line segment between two vertices
- *Planar* graph: has an embedding where no two edges cross
  - One graph may have many embeddings

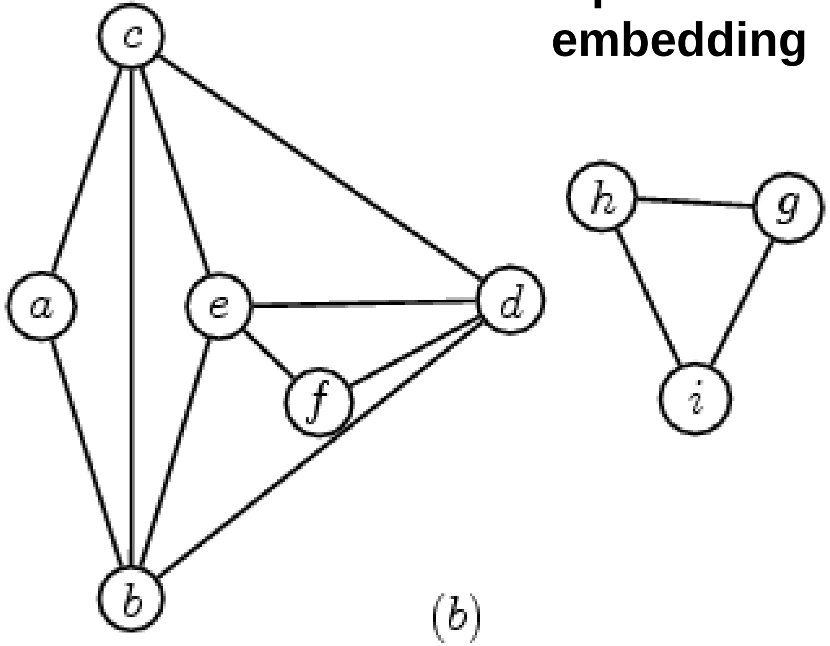
Graph and embedding example



(a)

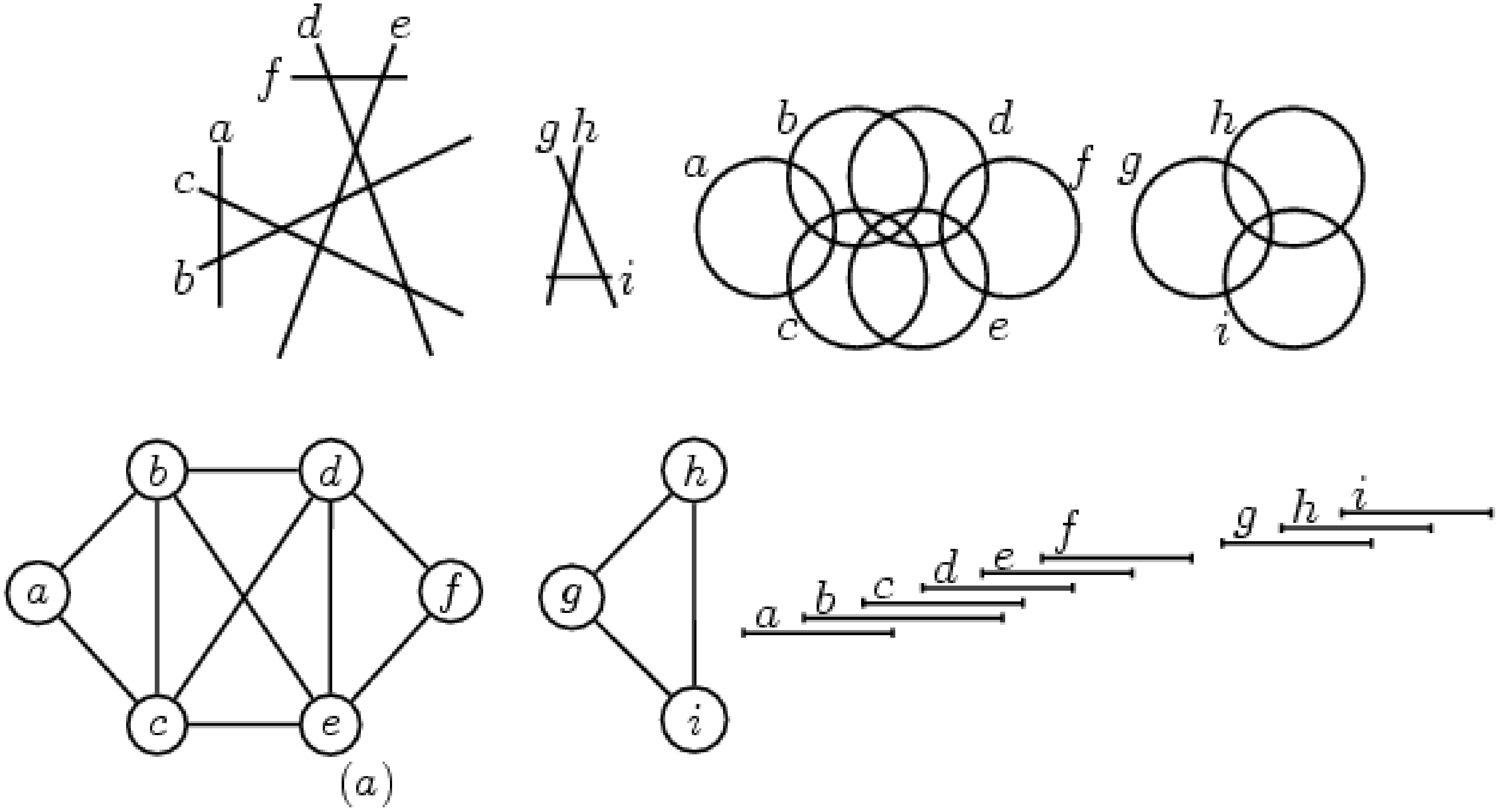
a non-planar embedding

a planar embedding



(b)

Other graph visualizations



Graph terminology

- Similar to digraphs
  - E.g. Adjacent vertices, path (length, simple path, simple cycle)
- An edge (u, v) is *incident* on u and v
- A path $\langle v_1, v_2, \ldots, v_n \rangle$ *connects* v_1 and v_n
- *Connected* graph: every pair of its vertices is connected
- *Subgraph* of a graph $G=(V, E)$: graph $G'=(V', E')$
  - $V' \subseteq V$
  - E' consists of edges $(v, w) \in E$ such that v and $w \in V'$
- *Induced subgraph*: E' consists of *all* edges $(v, w) \in E$ such that v and $w \in V'$
- *Free tree*: a connected acyclic graph. Properties:
  1. **Every free tree with $n \geq 1$ vertices contains exactly $n-1$ edges.**
  2. **If we add any edge to a free tree, we get a cycle**

Free tree properties

- Proof: (1) (i.e. $n \geq 1$ vertices $\Rightarrow$ $n-1$ edges) by induction with smallest counter example:
  - Suppose $G = (V, E)$ is a counter-example to (1) with the fewest vertices, $n=|V|$ vertices.
  - For $n \leq 1$ (the only free tree on one vertex has $|E|=0$), and thus $n > 1$.
  - No vertex can have zero incident edges (G would not be connected)
  - Suppose every vertex has *at least two edges incident*.
  - Start at $v_{1.}$. At each step, leave a vertex by a different edge from the one used to enter it => a path v_1, v_2, v_3,…

Free tree properties. Proof (cont'd)

- $|V| \neq \infty \Rightarrow \exists\, v_i = v_j$ for some $i < j;\ i \neq j - 1$ (there are no loops from a vertex to itself), $i \neq j - 2$ (else we entered and left vertex v_{i+1} on the same edge)
- Thus $i \leq j - 3$, and we have a cycle $v_i, v_{i+1}, v_j = v_i \Rightarrow$ we have contradicted the hypothesis that G had no vertex with only one edge incident => such a vertex v with edge (v, w) exists
- Consider the graph G' formed by deleting vertex v and edge (v, w) from G.
- G' cannot contradict (1) (if it did, it would be a smaller counter-example than G) => $|V| = n-1$ and $|E| = n-2$

Free tree properties. Proof (cont'd)

  - But ***G*** has $|V|=|V'|+1$ and $|E|=|E'|+1$ => ***G*** has ***n*-1** edges (proving that ***G*** does indeed satisfy (1))
  - No smallest counter-example to (1); we conclude there can be no counter-example at all, so (1) is true.
- For (2) (adding an edge to a free tree forms a cycle)
  - Assume it does not for a cycle=> adding the edge to a free tree of ***n*** vertices would be a graph with ***n*** vertices and ***n*** edges, connected, and we supposed that adding the edge left the graph acyclic. Thus we would have a free tree whose vertex and edge count did not satisfy condition (1)(i.e. contains exactly $n–1$ edges)

Methods of representation for graphs

- *Adjacency lists*: An array $\boldsymbol{Adj}$ of $|V|$ where $\boldsymbol{Adj[v]}$ is a set of all vertices adjacent to $\boldsymbol{v}$. Typically, this set is represented as a singly linked list.
- *Adjacency matrix*: An array $\boldsymbol{A}$ of size $|V| \times |V|$ with

$$\boldsymbol{A[i, j]} = 1 \text{ if } \boldsymbol{(i, j)} \in \boldsymbol{E}$$

$$\boldsymbol{A[i, j]} = 0 \text{ if } \boldsymbol{(i, j)} \notin \boldsymbol{E}$$

assuming that $\boldsymbol{V} = \{\mathbf{1, 2, \ldots,} \boldsymbol{n}\}$ (by renaming).

NOTE. For digraphs we could also use an:

- *Incidence matrix*: An array $\boldsymbol{B}$ of size $|V| \times |E|$ with

$$\boldsymbol{B[i. j]} = -1 \text{ if edge } j \text{ leaves vertex } i$$

$$\boldsymbol{B[i, j]} = 1 \quad \text{if edge } j \text{ enters vertex } i$$

$$\boldsymbol{B[i, j]} = 0 \quad \text{otherwise.}$$

Sparse and dense graphs

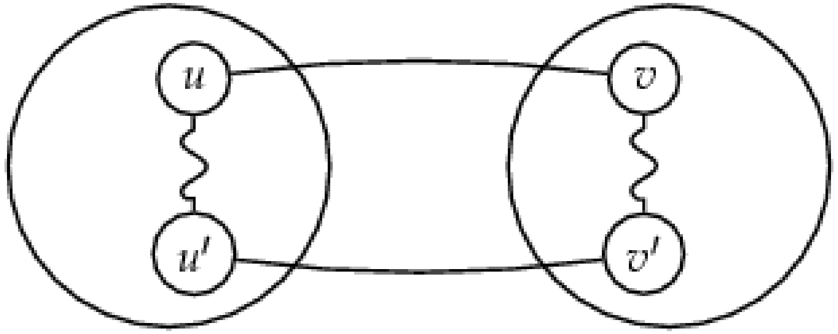
- A graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ is <u>sparse</u> if $|\boldsymbol{E}| \ll |\boldsymbol{V}|^2$.
- A graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ is <u>dense</u> if $|\boldsymbol{E}| \approx |\boldsymbol{V}|^2$.
- *Adjacency-lists* are preferred if the graph is *sparse*, because it is more compact – most algorithms assume adjacency-lists.
- *Adjacency-matrix* is preferred if the graph is *dense* or if the test whether two vertices are adjacent has to be fast – some algorithms depend on this fast tests.

The minimum cost spanning tree (MST)

- Spanning tree: is a free tree that connects all the vertices in V
  - cost of a spanning tree = sum of the costs of the edges in the tree
- Minimum spanning tree property:
  - $G = (V, E)$: a connected graph with a cost function defined on the edges; $U \subseteq V$.
  - If (u, v) is an edge of lowest cost such that $u \in U$ and $v \in V \setminus U$, then there is a minimum-cost spanning tree that includes (u, v) as an edge.

MST property. Proof (by contradiction)

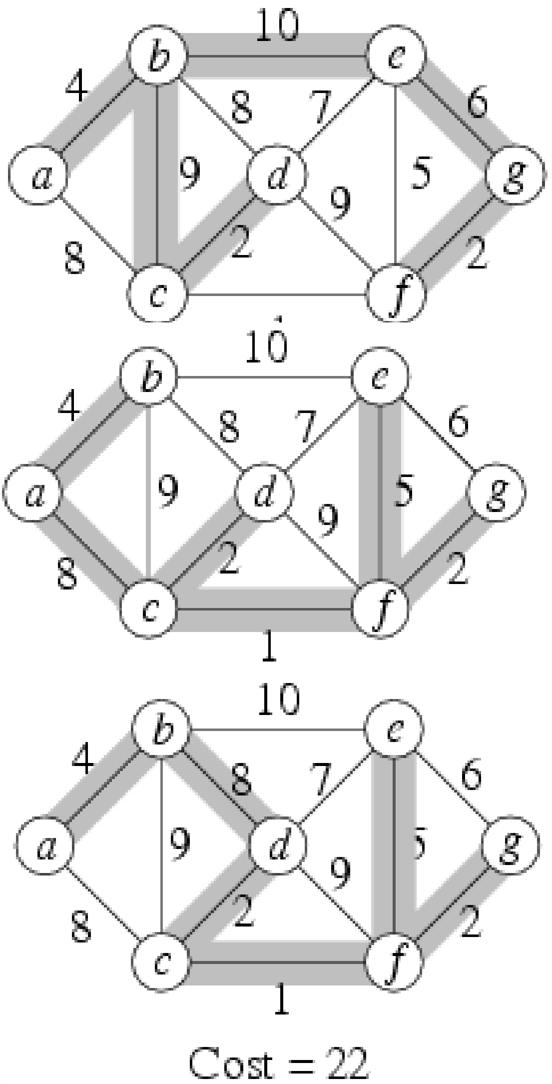
- Suppose that there is no minimum-cost spanning tree for G that includes (u, v)
- T: any minimum-cost spanning tree for G
- Adding (u, v) to T must introduce a cycle (T : a free tree and therefore satisfies property (2) for free trees). That cycle involves edge (u, v).
- Thus, there must be another edge (u', v') in T such that $u' \in U$ and $v' \in V \setminus U$
- Deleting the edge (u', v') breaks the cycle and yields a spanning tree T' whose cost $\leq$ cost of T (by assumption



$$c(u, v) \leq c(u', v'))$$

contradiction

MST

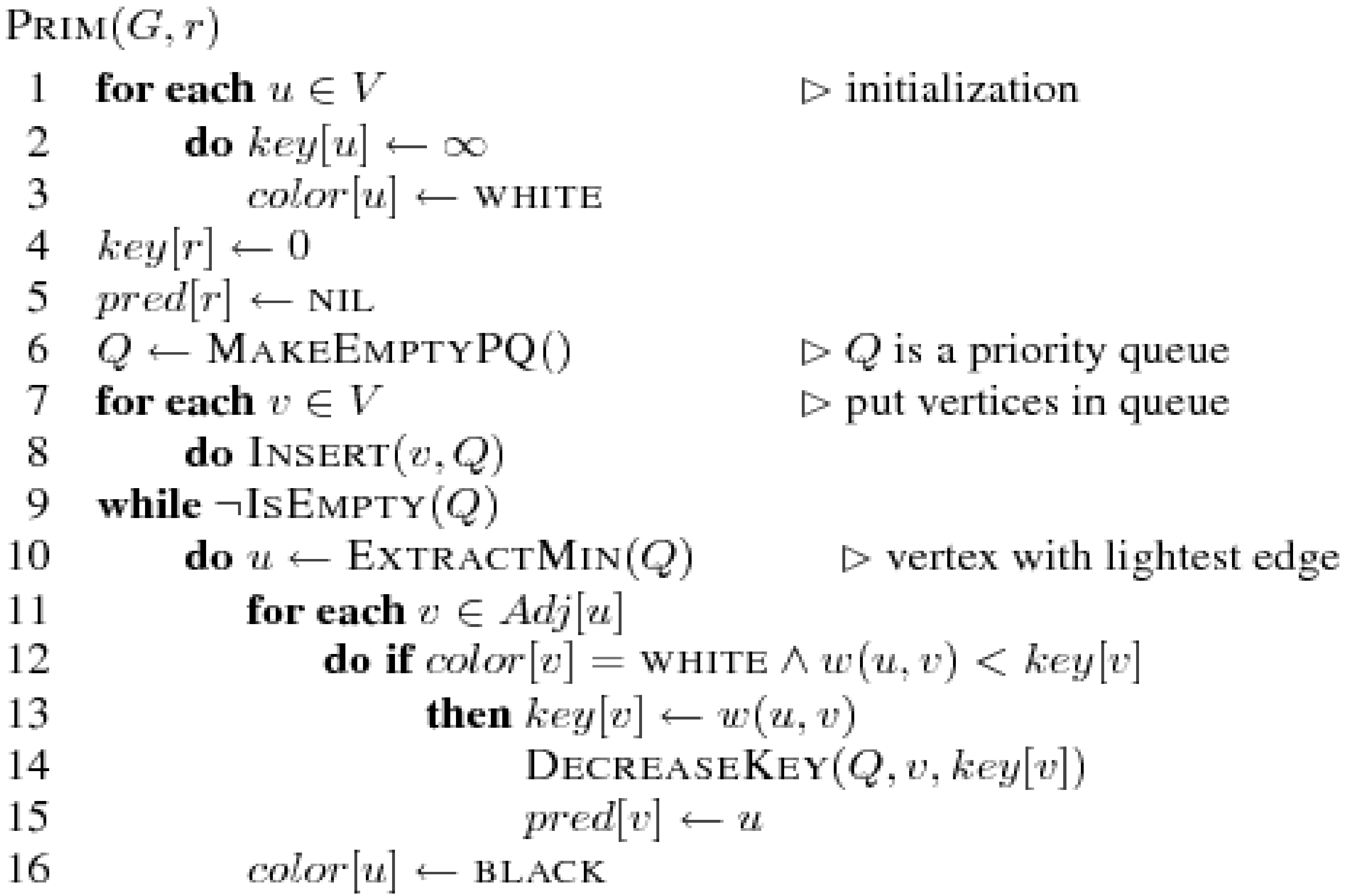


- Generic algorithm:
  - Maintains an acyclic subgraph *F of graph* G (*an intermediate spanning forest*)
  - Every component of *F*: minimum spanning tree of its vertices
  - Initially *F* consists of *n* one-node trees
  - The algorithm merges trees by adding certain edges between them
  - At halt, *F* consists of a single *n*-node tree, a MST
  - Two kinds of edges:
    - *Useless*: edge $\notin F$, both endpoints in same component of *F*
    - *Safe*: min weight edge with exactly one endpoint in a component
    - *Undecided*: rest of edges

Prim’s (Jarnik’s) algorithm

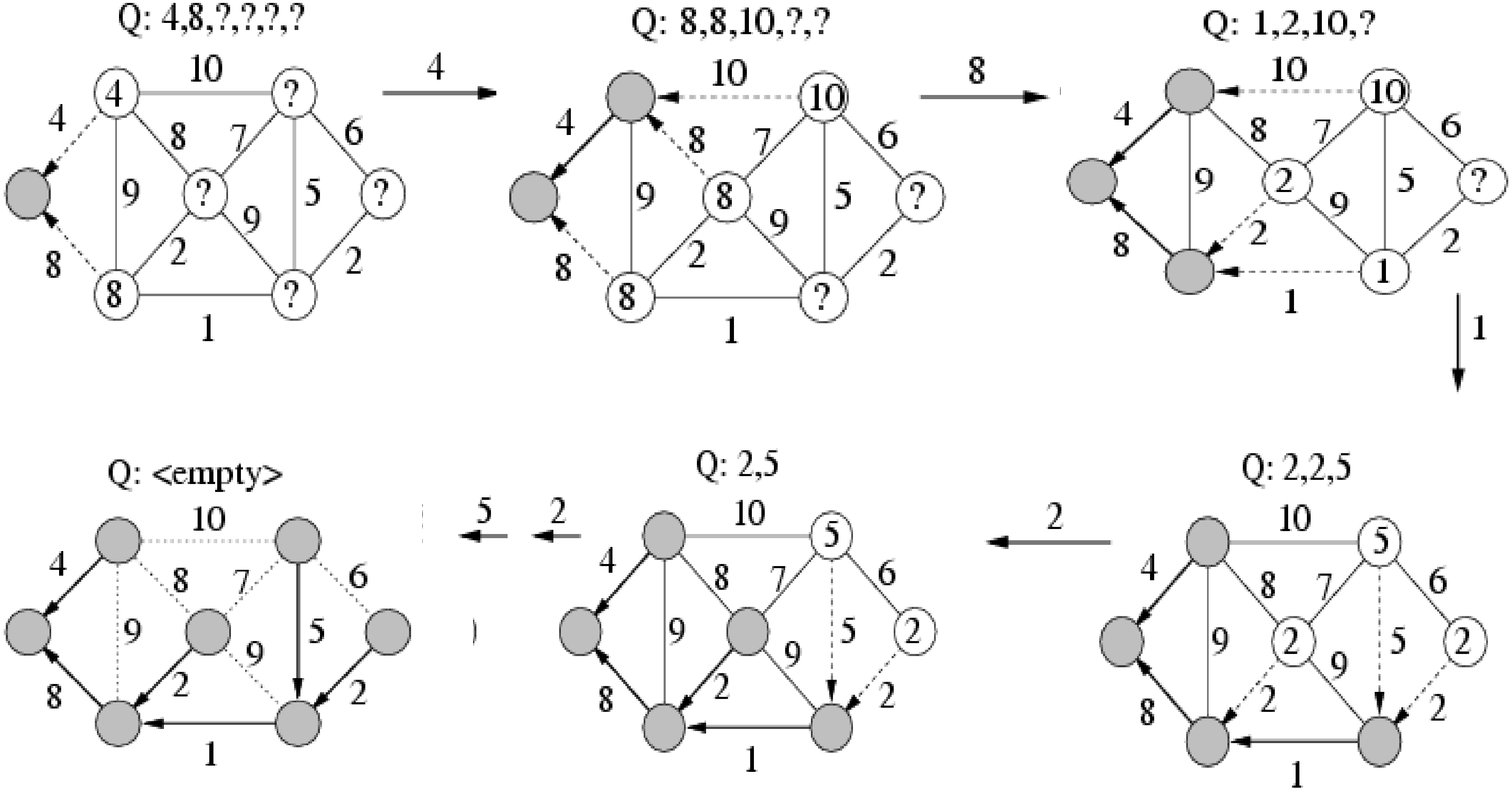
- Owed to Jarnik (1930), rediscovered by Prim (1956) and Dijkstra (1958)
- Algorithm:
  - Initially *T (*the only non-trivial component of *F*) contains an arbitrary vertex
  - Repeats find *T* ’s safe edge and add it to *T*
- Implementation:
  - Keep all edges adjacent to *T* in heap *Q*
  - Extract minimum edge and check if both endpoints in *T* (by checking its color)
  - If not add new edge to *T* and add new adjacent edges to heap

MST. Prim’s algorithm



```
PRIM($G, r$)
for each $u \in V$                                   ▷ initialization
    do $key[u] \leftarrow \infty$
       $color[u] \leftarrow$ WHITE
$key[r] \leftarrow 0$
$pred[r] \leftarrow$ NIL
$Q \leftarrow$ MAKEEMPTYPQ()                          ▷ $Q$ is a priority queue
for each $v \in V$                                   ▷ put vertices in queue
    do INSERT($v, Q$)
while $\neg$ISEMPTY($Q$)
    do $u \leftarrow$ EXTRACTMIN($Q$)                  ▷ vertex with lightest edge
       for each $v \in Adj[u]$
           do if $color[v] =$ WHITE $\wedge w(u, v) < key[v]$
                 then $key[v] \leftarrow w(u, v)$
                      DECREASEKEY($Q, v, key[v]$)
                      $pred[v] \leftarrow u$
       $color[u] \leftarrow$ BLACK
```

Prim’s algorithm example

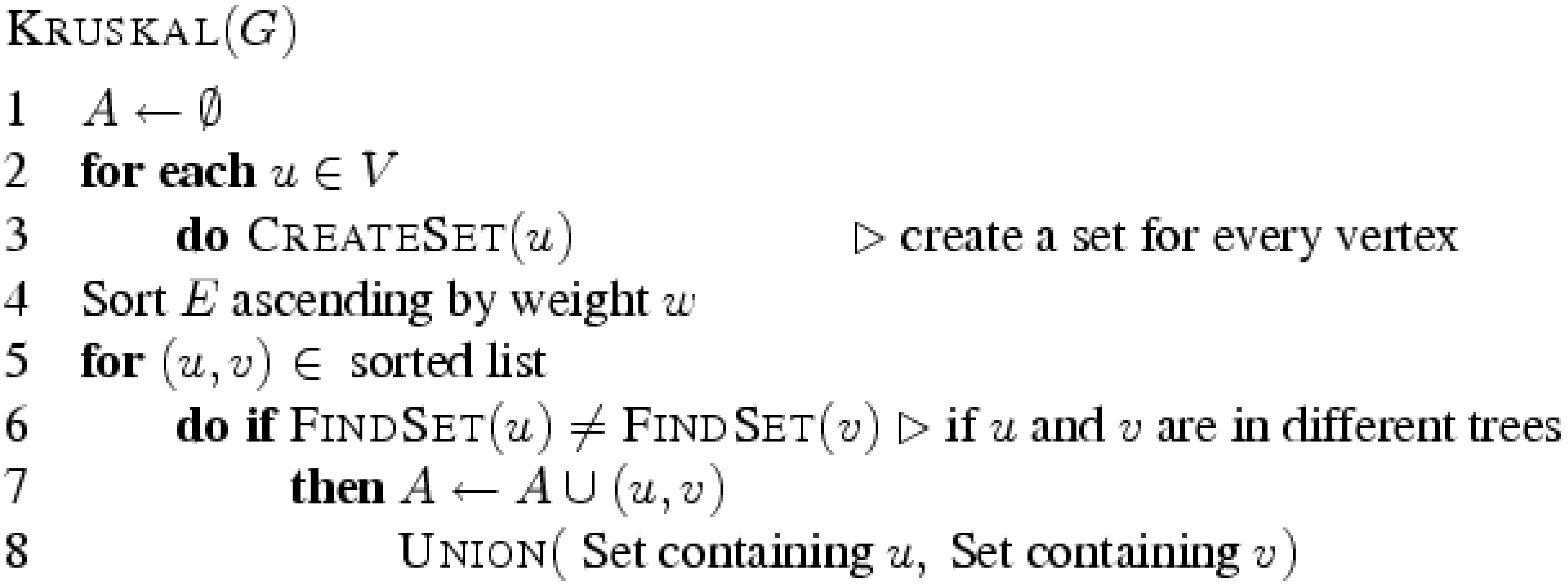


MST. Prim’s algorithm

- Running time
  - Dominated by the cost of the heap operations: insert, extractMin and DecreaseKey
    - Insert and extractMin are called $\boldsymbol{O}(|\boldsymbol{V}|=\boldsymbol{n})$ times (once per vertex, except $\boldsymbol{r}$)
  - operations mentioned before can be performed in $O(\log n)$ time, for a heap of n items

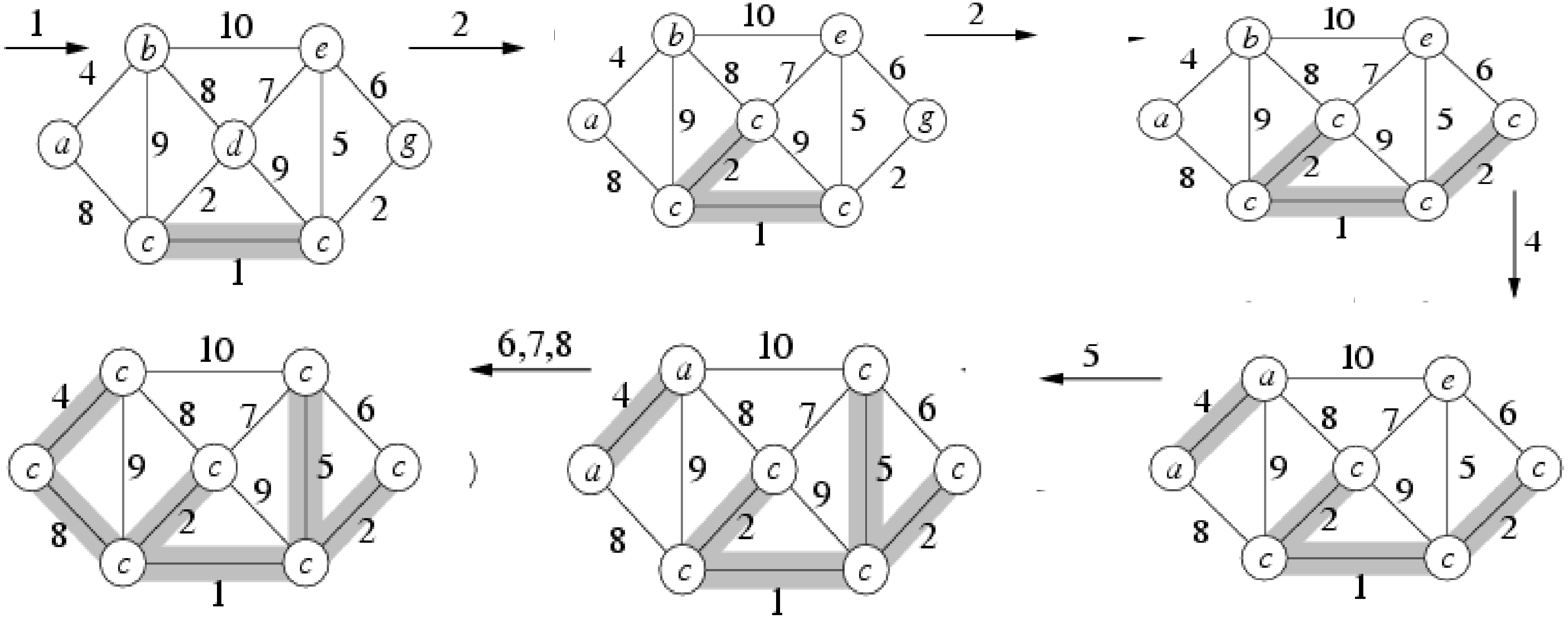
MST. Kruskal's algorithm

- Owed to Kruskal (1956)
- Examine all edges in increasing weight order
  - Any edge we examine is safe



```
KRUSKAL($G$)
$A \leftarrow \emptyset$
for each $u \in V$
        do CREATESET($u$)                    ▷ create a set for every vertex
Sort $E$ ascending by weight $w$
for $(u, v) \in$ sorted list
        do if FINDSET($u$) $\neq$ FINDSET($v$) ▷ if $u$ and $v$ are in different trees
              then $A \leftarrow A \cup (u, v)$
                    UNION( Set containing $u$, Set containing $v$ )
```

Kruskal's algorithm example



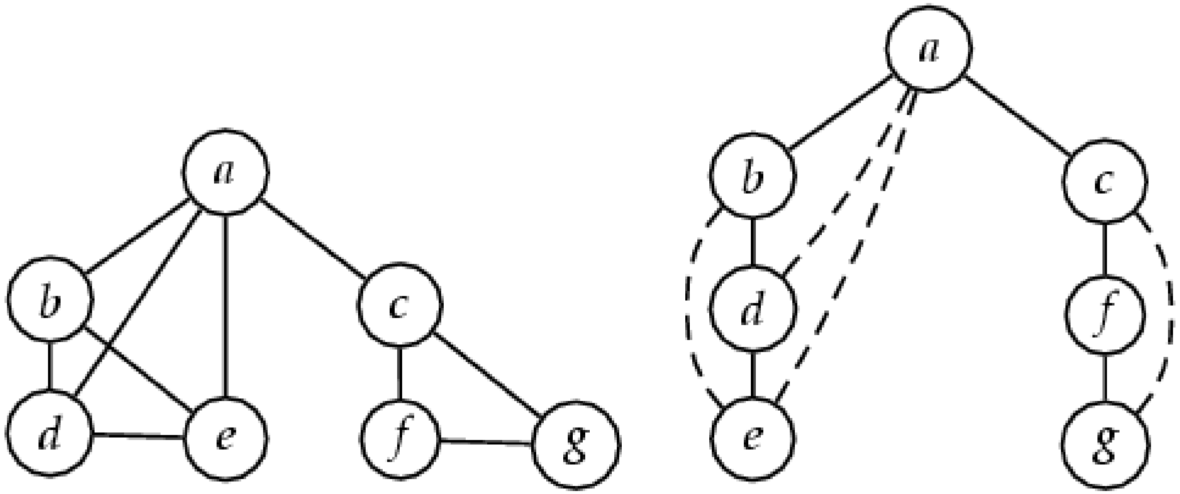
c-f	*c-d*	*f-g*	*a-b*	*e-f*	*e-g*	*d-e*	*a-c*	*b-d*	*b-c*	*d-f*	*b-e*
1	2	2	4	5	6	7	8	8	9	9	10

Animations of Prim's and Kruskal's Algorithms

- http://www.unf.edu/~wkloster/foundations/PrimApplet/PrimApplet.htm
- http://students.ceid.upatras.gr/%7Epapagel/project/prim.htm
- http://www.math.ucsd.edu/~fan/algo/CS101.swf

Traversals. Depth-First Search

- depth-first spanning forests constructed for undirected graphs are even simpler than for digraphs
    - each tree in the forest is one connected component of the graph => if a graph is connected, it has only one tree in it's depth-first spanning forest
    - only two kinds of edges: tree edges and back (forward) edges

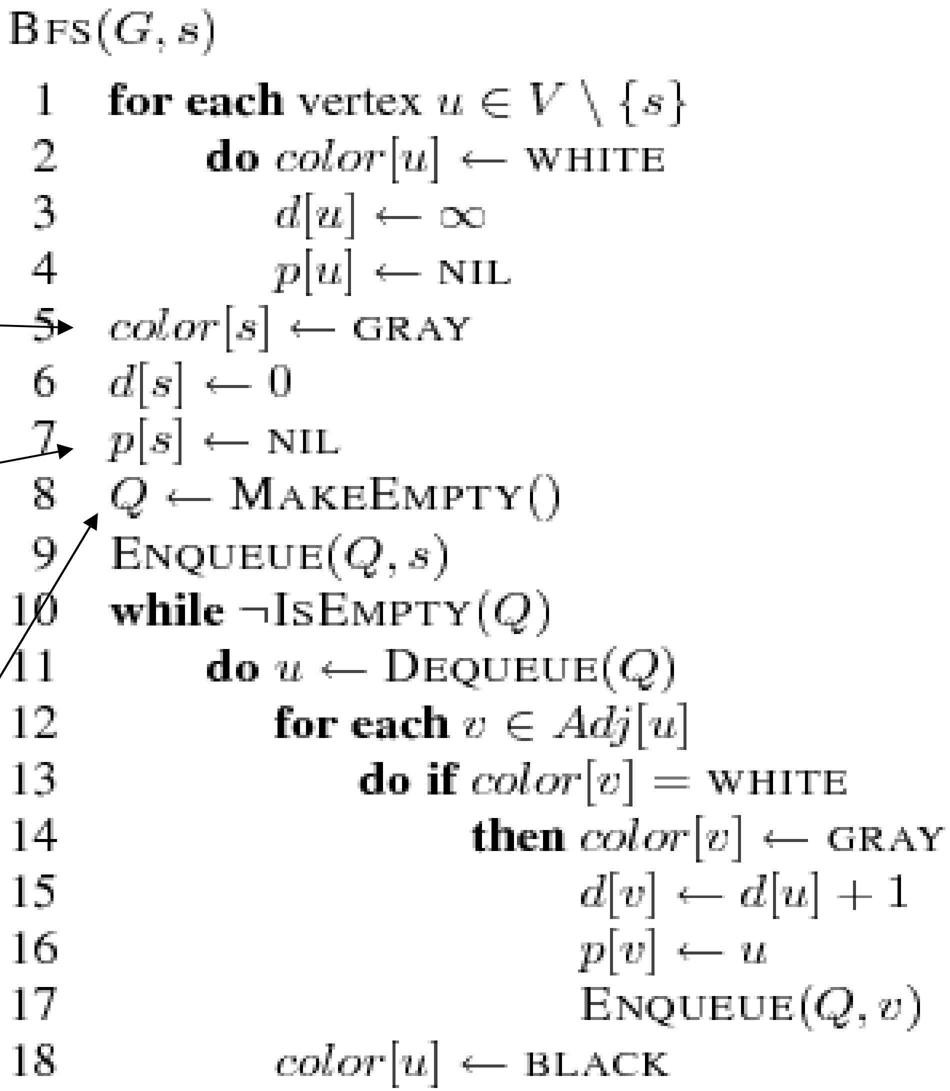


Breadth-First Search (BFS)

- The BFS algorithm computes the distance (length of the shortest path) of a start vertex s to <u>all reachable</u> vertices.
- Breadth-first search expands the frontier between discovered and undiscovered vertices uniformly across the frontier:
  - it discovers all vertices with distance k from s before discovering vertices at $k + 1$.
- Breadth-first search colors vertices white, gray, or black:
  - all undiscovered vertices are white;
  - discovered vertices on the frontier are gray;
  - discovered vertices not on the frontier are black.
  - Thus black and white vertices can never be adjacent.

… Breadth-First Search

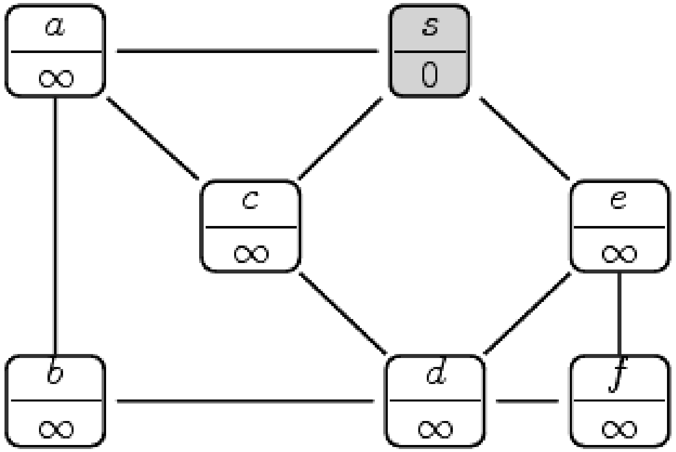
- **Explores neighborhood of a vertex**
- **Marks vertices using array *color***
- **Maintains the predecessors (parents) of each vertex in *p* and its distance to *s* in *d***
- **Uses a queue to keep vertices while processing**



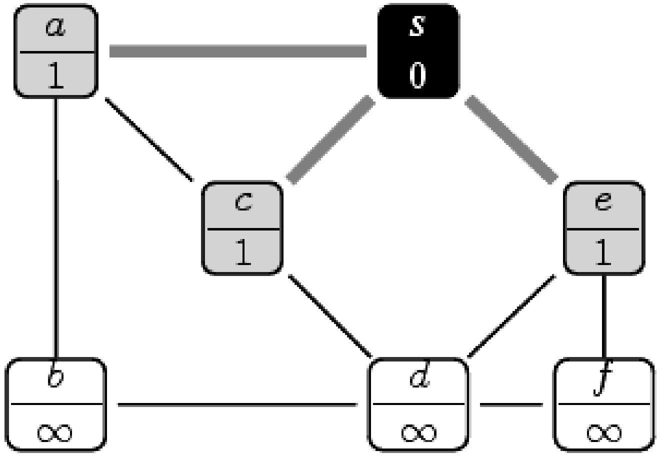
```
BFS(G, s)

for each vertex $u \in V \setminus \{s\}$
    do $color[u] \leftarrow$ WHITE
       $d[u] \leftarrow \infty$
       $p[u] \leftarrow$ NIL
$color[s] \leftarrow$ GRAY
$d[s] \leftarrow 0$
$p[s] \leftarrow$ NIL
$Q \leftarrow$ MAKEEMPTY()
ENQUEUE($Q, s$)
while ¬ISEMPTY($Q$)
    do $u \leftarrow$ DEQUEUE($Q$)
       for each $v \in Adj[u]$
           do if $color[v] =$ WHITE
                 then $color[v] \leftarrow$ GRAY
                      $d[v] \leftarrow d[u] + 1$
                      $p[v] \leftarrow u$
                      ENQUEUE($Q, v$)
       $color[u] \leftarrow$ BLACK
```

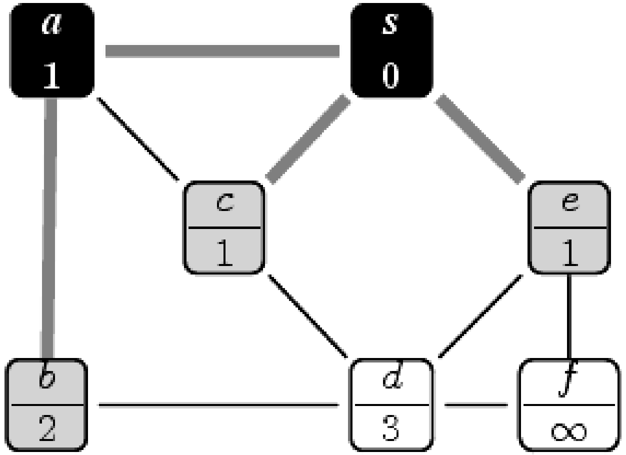
BFS example



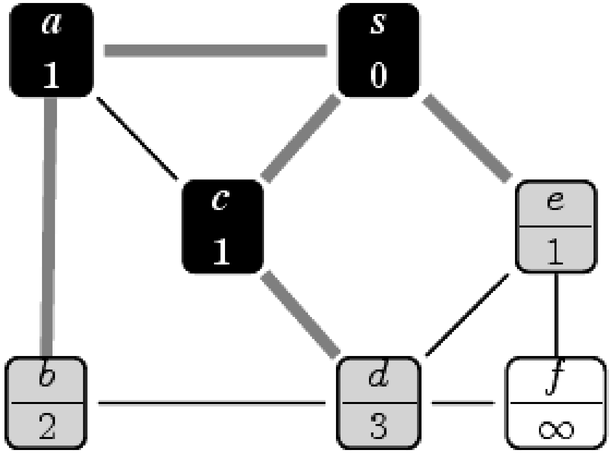
Q: s



Q: a, c, e

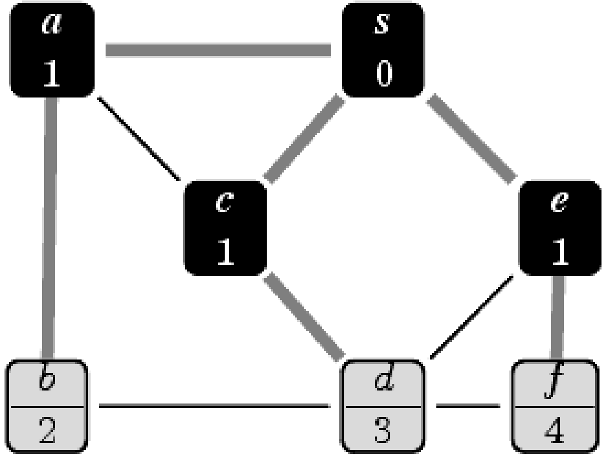


Q: c, e, b

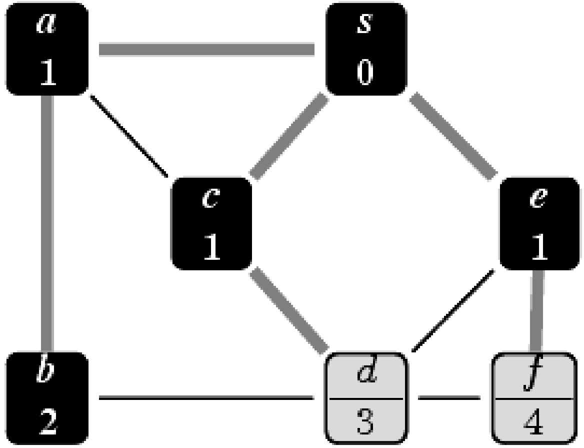


Q: e, b, d

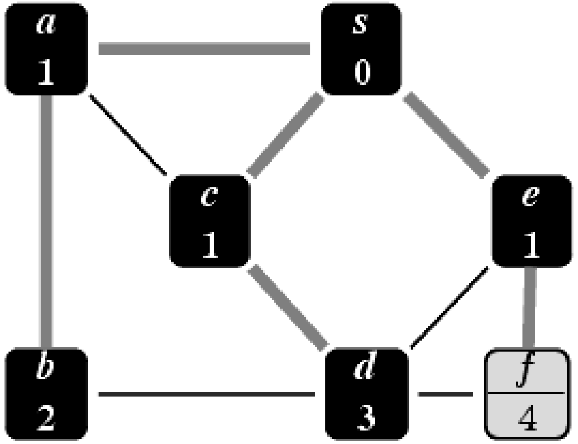
BFS example



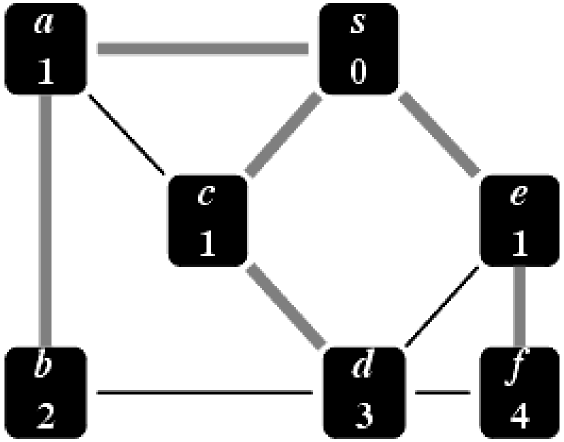
Q: b, d, f



Q: d, f



Q: f



Q: Empty

Analysis of BFS

- The initialization takes $O(n)$.
- Inner loop: each the adjacency list of each vertex is scanned at most once.
    - Since the sum of the length of all adjacency lists is $|E|$, the main loop takes $O(e)$
- Total running time is therefore $O(n + e)$.

- Animations: http://www.cs.duke.edu/csed/jawaa2/examples/BFS.html

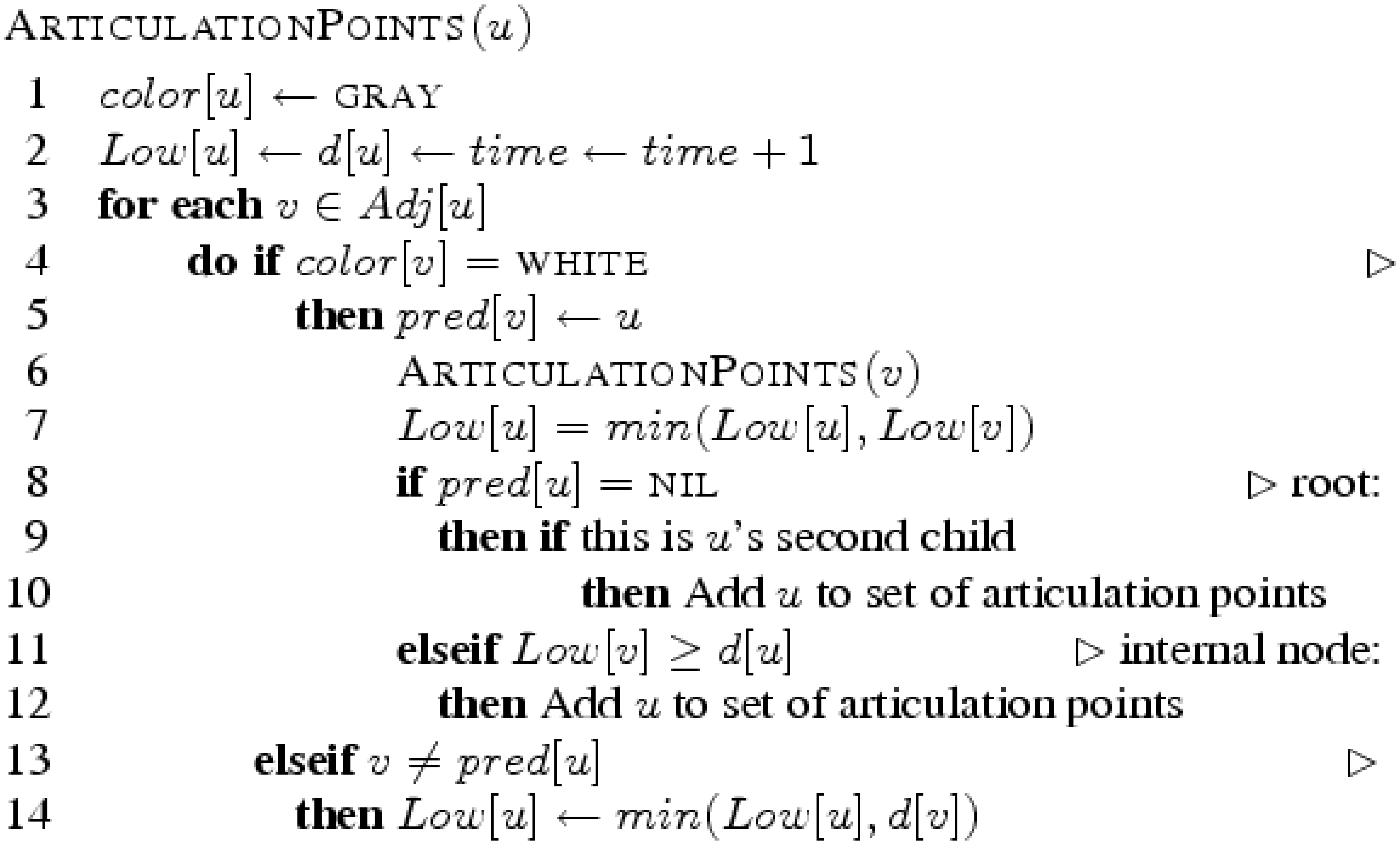
Articulation Points and Bi-connected Components

- Both DFS and BFS can be used to find connected components (they are trees of either spanning forest)
- Articulation point of a graph: a vertex v such that when we remove v and all edges incident upon v we break a connected component of the graph into two or more pieces
- Bi-connected graph: connected graph with no articulation points
- Can use DFS to find articulation points

Articulation Points...

- Algorithm to find all the articulation points of a connected graph
  - Perform a DFS of the graph, computing ***dfnumber*[*v*]** (or discovery times, ***d* [*v*]**])
  - For each vertex ***v***, compute ***low***[***v***], which is the smallest ***dfnumber*** of ***v*** or of any vertex ***w*** reachable from *v* by visiting the vertices in a postorder traversal.
  - ***low***[***v***]= **min**(***d***[***v***], ***d***[***z***](∃ back edge (***v*, *z***)), ***low***[***y***](∀ ***y*** child of ***v***))
  - Find the articulation points as follows:
    - The root is an articulation point iff it has two or more children
    - A vertex ***v*** other than the root is an articulation point iff ∃ child ***w*** of ***v*** such that ***low***[***w***] ≥ ***d***[***v***].
  - Following algorithm used instead of the DFSVisit part

Articulation Points...



```
ARTICULATIONPOINTS($u$)
$color[u] \leftarrow$ GRAY
$Low[u] \leftarrow d[u] \leftarrow time \leftarrow time + 1$
for each $v \in Adj[u]$
    do if $color[v] =$ WHITE                                   ▷
        then $pred[v] \leftarrow u$
            ARTICULATIONPOINTS($v$)
            $Low[u] = min(Low[u], Low[v])$
            if $pred[u] =$ NIL                                 ▷ root:
              then if this is $u$'s second child
                      then Add $u$ to set of articulation points
            elseif $Low[v] \geq d[u]$                          ▷ internal node:
              then Add $u$ to set of articulation points
    elseif $v \neq pred[u]$                                    ▷
        then $Low[u] \leftarrow min(Low[u], d[v])$
```

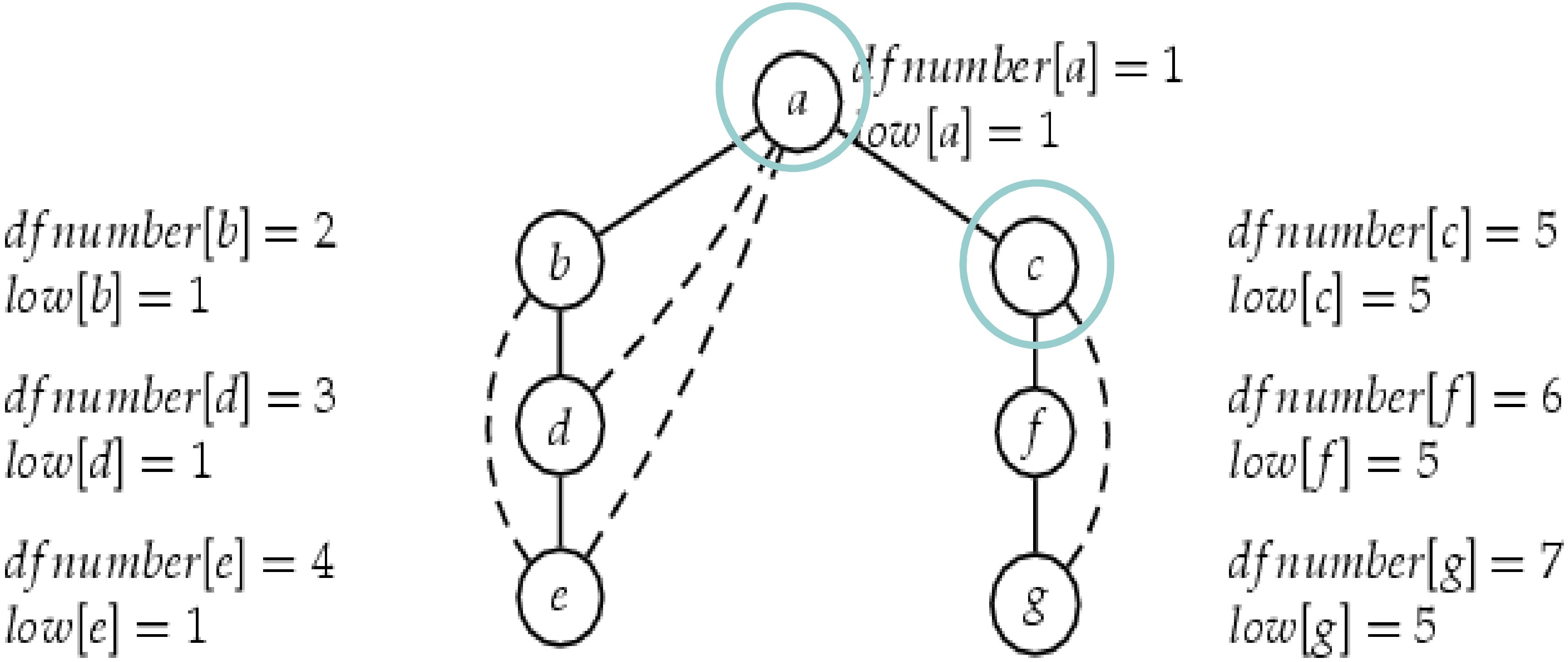
Reminder. Depth first search

```
DFS($G$)
for each $u \in V$                              ▷ initialization
    do $color[u] \leftarrow$ WHITE
       $pred[u] \leftarrow$ NIL
$time \leftarrow 0$;
for each $u \in V$
    do if $color[u] =$ WHITE                    ▷ found an undiscovered vertex
        then DFSVisit($u$)                      ▷ start a new search here

DFSVisit($u$)
$color[u] \leftarrow$ GRAY                      ▷ mark $u$ visited
$d[u] \leftarrow time \leftarrow time + 1$
for each $v \in Adj[u]$
    do if $color[v] =$ WHITE                    ▷ if neighbor $v$ undiscovered
        then $pred[v] \leftarrow u$             ▷ ...set predecessor pointer
             DFSVisit($v$)                      ▷ ...visit $v$
$color[u] \leftarrow$ BLACK                     ▷ we're done with $u$
$f[u] \leftarrow time \leftarrow time + 1$
```

Articulation points example

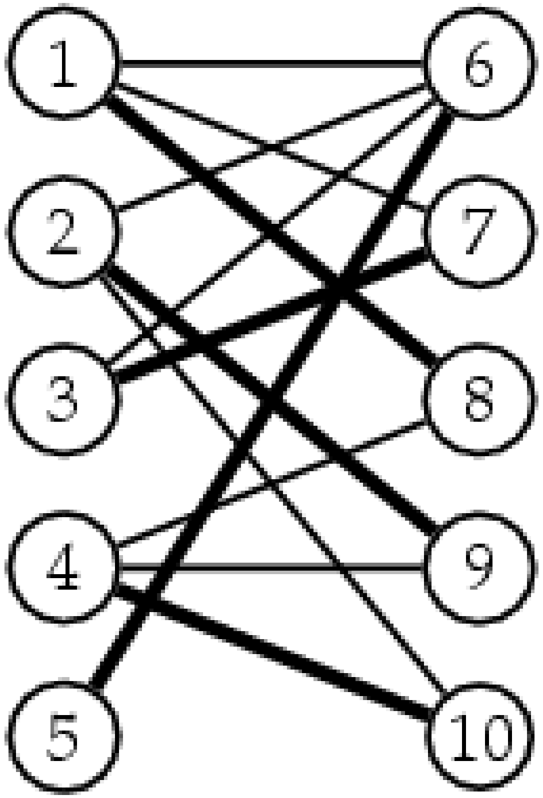
- Note: back edges are dashed lines
- Articulation point are circled



Graph matching

- *Bipartite graph*: its vertices can be divided into two disjoint groups with each edge having one end in each group
- *Matching*: given a graph $\boldsymbol{G}=(\boldsymbol{V}, \boldsymbol{E})$, a subset of the edges in $\boldsymbol{E}$ with no two edges incident upon the same vertex in $\boldsymbol{V}$
- *Maximum cardinality matching problem*: the task of selecting a maximum subset of such edges (maximal matching)
- *Complete matching*: matching in which every vertex is an endpoint of some edge in the matching

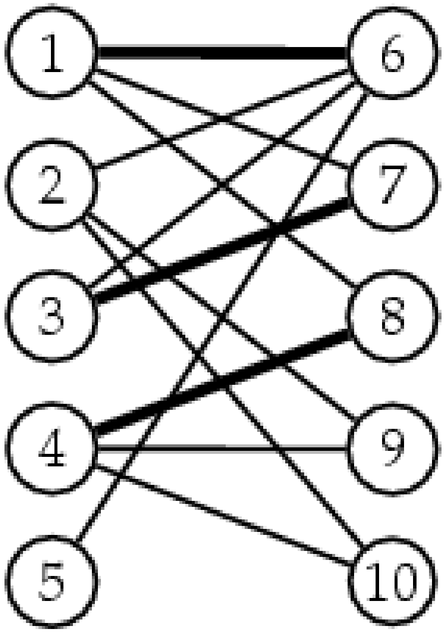
Graph matching



A bipartite graph and a matching

- Algorithms:
  - Brute force: systematically generate all matchings and select largest – exponential time
  - More efficient: use *augmenting paths*
  - *Augmenting path relative to* ***M****:*
    - ***M***:*a* matching in a graph ***G.***
    - Vertex ***v*** is matched if it is the endpoint of an edge in ***M.***
    - *Augmenting path:* a path connecting two unmatched vertices in which alternate edges in the path are in ***M*** .

Matching and augmenting path example



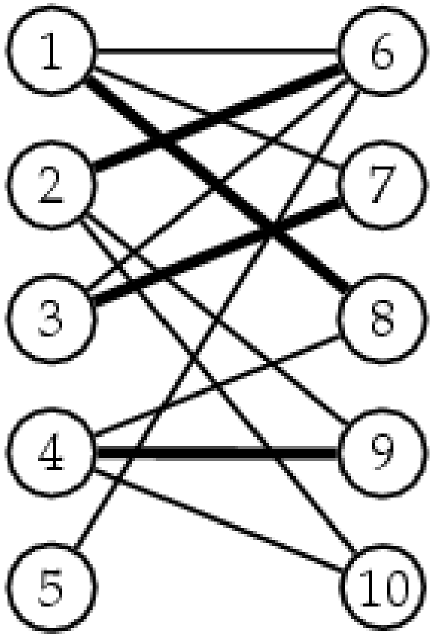
(a) Matching

(b) Augmenting path

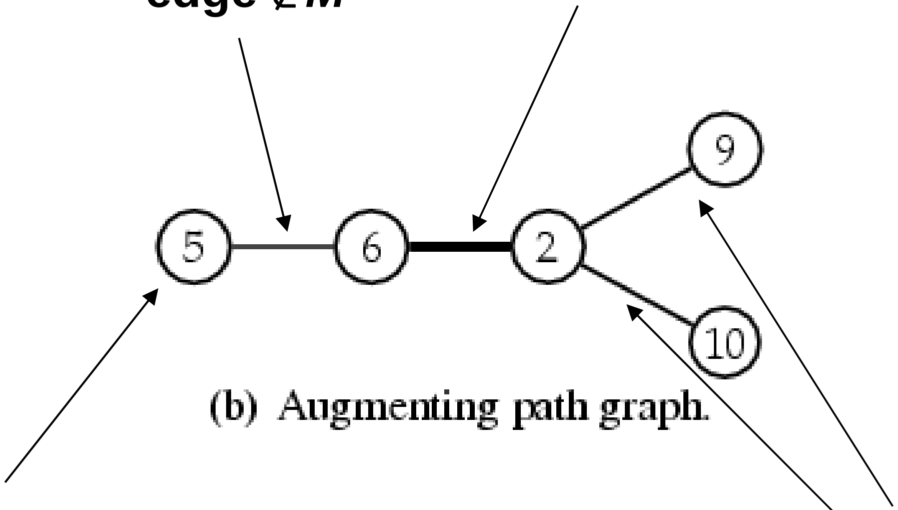
Maximal matching

```
MaxMatching($G$)
$M \leftarrow \phi$
repeat
        Find an augmenting path $P$ relative to $M$
        $M \leftarrow M \setminus P$
    until no further augmenting paths exist
return $M$
```

- **Finding an augmenting path (similar to BFS)**
  - ***G*: bipartite graph with vertices partitioned into sets V_1 and V_2**
  - **Level 0:all unmatched vertices from V_i.**
  - **At odd level *i* , add new vertices that are adjacent to a vertex at level *i*-1, by a non-matching edge, and we also add that edge.**
  - **At even level *i*, add new vertices that are adjacent to a vertex at level *i*-1 because of an edge in the matching *M*, together with that edge.**
  - **Repeat until an unmatched vertex is added at an odd level, or until no more vertices can be added**



The larger matching.



(b) Augmenting path graph.

Level 1:
edge $\notin M$

Level 2:
edge $\in M$

Unmatched
vertex
at level 0

Level 3:
either
edge $\notin M$

Matching – Hopcroft-Karp Algorithm

- $G = U + V$ (the two sets in the bipartition of)
- *M* is *a* matching from *U* to *V* at any time.
- The algorithm is run in *phases*. Each phase consists of the following steps.
  - A breadth first search partitions the vertices of the graph into layers.
    - The free vertices in *U* are used as the starting vertices of this search, and form the first layer of the partition.
    - At the first level of the search, only unmatched edges may be traversed (since *U* is not adjacent to any matched edges);
    - at subsequent levels of the search, the traversed edges are required to alternate between unmatched and matched. That is, when searching for successors from a vertex in *U*, only unmatched edges may be traversed, while from a vertex in *V* only matched edges may be traversed.
    - The search terminates at the first layer *k* where one or more free vertices in *V* are reached.

Matching – Hopcroft-Karp Algorithm

  - All free vertices in *V* at layer *k* are collected into a set *F*. That is, a vertex *v* is put into F if and only if it ends a shortest augmenting path.
  - The algorithm finds a maximal set of *vertex disjoint* augmenting paths of length *k*.
    - set may be computed by depth first search from *F* to the free vertices in *U*, using the breadth first layering to guide the search:
    - the depth first search is only allowed to follow edges that lead to an unused vertex in the previous layer, and
    - paths in the depth first search tree must alternate between unmatched and matched edges.
    - Once an augmenting path is found that involves one of the vertices in *F*, the depth first search is continued from the next starting vertex.
- Every one of the paths found in this way is used to enlarge *M*.
- The algorithm terminates when no more augmenting paths are found in the breadth first search part of one of the phases

Matching – Hopcroft-Karp Algorithm

```
/*
  G = G1  U G2 U {NIL}
  where G1 and G2 are partition of
graph
  and NIL is a special null vertex
*/
function BFS ()
  for v in G1
    if Pair[v] == NIL
      Dist[v] = 0
      Enqueue(Q,v)
    else
      Dist[v] = inf
  Dist[NIL] = inf
  while Empty(Q) == false
    v = Dequeue(Q)
    if v != NIL
      for each u in Adj[v]
        if Dist[ Pair[u] ] == inf
          Dist[ Pair[u] ] =
             Dist[v] + 1
          Enqueue(Q,Pair[u])
  return Dist[NIL] !=inf
```

```
function DFS (v)
  if v != NIL
    for each u in Adj[v]
      if Dist[ Pair[u] ] == Dist[v] + 1
        if DFS(Pair[u]) == true
          Pair[u] = v
          Pair[v] = u
          return true
    Dist[v] = inf
    return false
  return true
function Hopcroft-Karp
  for each v in G
    Pair_G1[v] = NIL
    Pair_G2[v] = NIL
  matching = 0
  while BFS() == true
    for each v in G1
      if Pair_G1[v] == NIL
        if DFS(v) == true
          matching = matching + 1
  return matching
```

The Marriage Problem and Matchings

- $G=(V,E)$ is a bipartite graph, with vertex classes X and Y;
- M is the current matching;
- *spouse[y]* is null, if y in Y is not currently matched; otherwise it is x, where xy is an edge in M.
- *color[v]* is WHITE, if v is an unmatched vertex, GREEN if it is matched, in the current matching;

```
// Initialization
for y in Y spouse[y] = null; // M is empty initially
for v in V color[v] = WHITE; // everybody unmatched
  do {
       search for an a-path in G,
       $x_0$, $y_0$, ..., $x_i$, $y_i$ ,
       where color[$x_0$] = color[$y_i$] = WHITE, and
       spouse[$y_j$] = $x_{j+1}$ for j = 0 .. i-1;
       if (there is no a-path in G) halt; otherwise
       // Improve matching
               for j = 0 .. i
                       spouse[$y_j$] = $x_j$;
           color[$x_0$] = GREEN; color[$y_i$] = GREEN;
}
```

Reading

- AHU, chapter 7
- Preiss, chapter: Graphs and Graph Algorithms
- CLR, chapter 23, section 2, chapter 24
- CLRS chapter 22, section 2, 3, chapter 26 section 3
- Notes