Algorithm Analysis

Correctness of Algorithms.

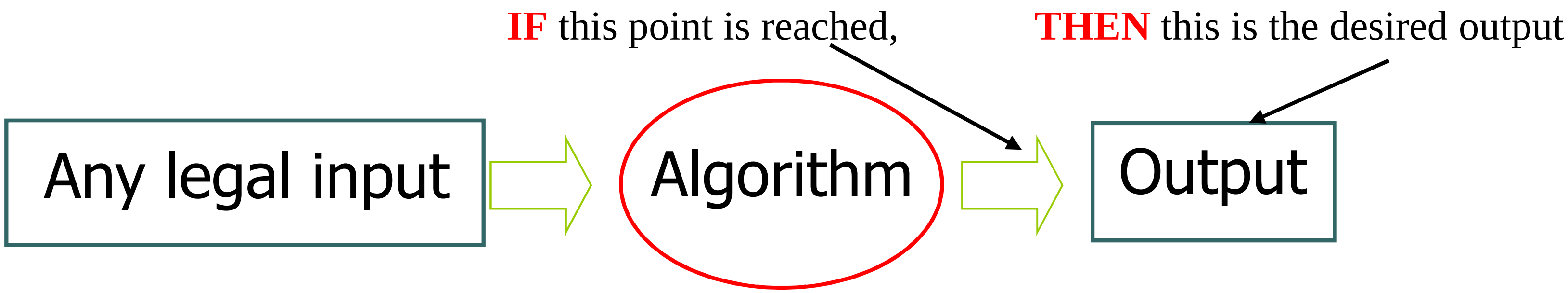
Efficiency of Algorithms

Correctness of Algorithms

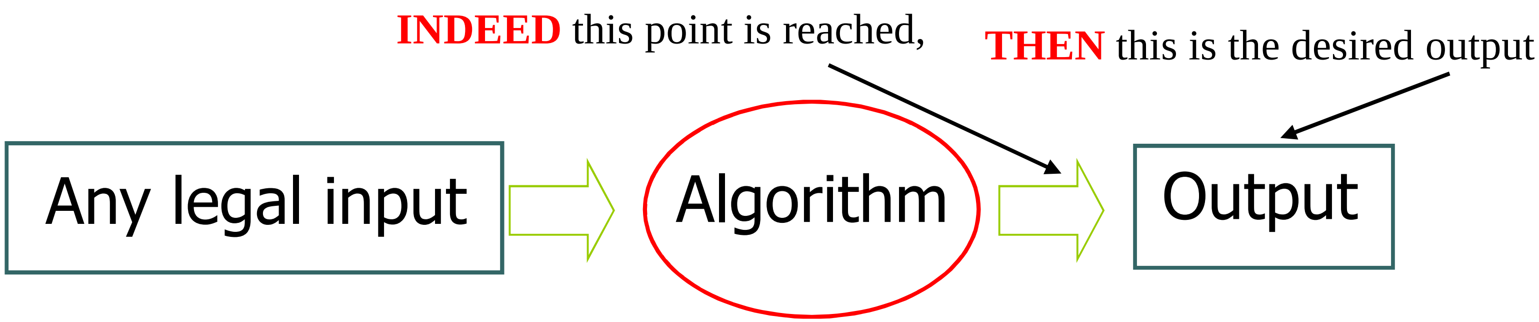
- An algorithm is *correct* if, for any legal input, it halts (terminates) with the correct output.
- A correct algorithm *solves* the given computational problem.
- Automatic proof of correctness is not possible
- But there are practical techniques and rigorous formalisms that help to reason about the correctness of algorithms

Partial and Total Correctness

- *Partial* correctness



- *Total* correctness



Assertions

- To prove partial correctness we associate a number of **assertions** (statements about the state of the execution) with specific checkpoints in the algorithm.
  - E.g., $A[1], \ldots, A[k]$ form an increasing sequence
- **Preconditions** – assertions that must be valid *before* the execution of an algorithm or a subroutine
- **Postconditions** – assertions that must be valid *after* the execution of an algorithm or a subroutine

Loop Invariants

- **Invariants** – assertions that are valid any time they are reached (many times during the execution of an algorithm, e.g., in loops)
- We must show three things about loop invariants:
  - **Initialization** – it is true prior to the first iteration
  - **Maintenance** – if it is true before an iteration, it remains true before the next iteration
  - **Termination** – when loop terminates the invariant gives a useful property to show the correctness of the algorithm

Example of Loop Invariants (1)

Invariant: *at the start of each **for** loop,* $A[1\ldots j-1]$ *consists of elements originally in* $A[1\ldots j-1]$ *but in sorted order*

```
for j=2 to length(A)
   do key=A[j]
      i=j-1
      while i>0 and A[i]>key
        do A[i+1]=A[i]
           i--
      A[i+1]:=key
```

Example of Loop Invariants (2)

Invariant: *at the start of each **for** loop,* $\boldsymbol{A[1\ldots j-1]}$ *consists of elements originally in* $\boldsymbol{A[1\ldots j-1]}$ *but in sorted order*

```
for j=2 to length(A)
   do key=A[j]
      i=j-1
      while i>0 and A[i]>key
        do A[i+1]=A[i]
           i--
      A[i+1]:=key
```

- **Initialization**: $j = 2$, the invariant trivially holds because $\boldsymbol{A[1]}$ is a sorted array

Example of Loop Invariants (3)

- **Invariant**: *at the start of each **for** loop,* $\mathbf{A}[1\ldots j-1]$ *consists of elements originally in* $\mathbf{A}[1\ldots j-1]$ *but in sorted order*

```
for j=2 to length(A)
   do key=A[j]
      i=j-1
      while i>0 and A[i]>key
        do A[i+1]=A[i]
           i--
      A[i+1]:=key
```

- **Maintenance**: the inner **while** loop moves elements $A[j-1]$, $A[j-2]$, …, $A[j-k]$ one position right without changing their order. Then the former $A[j]$ element is inserted into *k*-th position so that $A[k-1] \leq A[k] \leq A[k+1]$.

$A[1\ldots j-1]$ sorted + $A[j] \longrightarrow A[1\ldots j]$ sorted

Example of Loop Invariants (3)

- **Invariant**: *at the start of each **for** loop, $A[1\ldots j-1]$ consists of elements originally in $A[1\ldots j-1]$ but in sorted order*

```
for j=2 to length(A)
   do key=A[j]
      i=j-1
      while i>0 and A[i]>key
        do A[i+1]=A[i]
           i--
      A[i+1]:=key
```

- **Termination**: the loop terminates, when $j = n+1$. Then the invariant states: "$A[1\ldots n]$ *consists of elements originally in* $A[1\ldots n]$ *but in sorted order"*

Summations

- The running time of insertion sort is determined by a nested loop

```
for j←2 to length(A)
   key←A[j]
   i←j-1
   while i>0 and A[i]>key
      A[i+1]←A[i]
      i←i-1
   A[i+1]←key
```

- Nested loops correspond to summations

$$\sum_{j=2}^{n}(j-1)=O(n^2)$$

Proof by Induction

- We want to show that property $\boldsymbol{P}$ is true for all integers $\boldsymbol{n} \geq \boldsymbol{n}_0$
- **Basis**: prove that $\boldsymbol{P}$ is true for $\boldsymbol{n}_0$
- **Inductive step**: prove that if $\boldsymbol{P}$ is true for all $\boldsymbol{k}$ such that $\boldsymbol{n}_0 \leq \boldsymbol{k} \leq \boldsymbol{n} - 1$ then $\boldsymbol{P}$ is also true for $\boldsymbol{n}$
- Example

$$S(n) = \sum_{i=0}^{n} i = \frac{n(n+1)}{2} \text{ for } n \geq 1$$

- Basis

$$S(1) = \sum_{i=0}^{1} i = \frac{1(1+1)}{2}$$

Proof by Induction (2)

- Inductive Step

$$S(k)=\sum_{i=0}^{k} i = \frac{k(k+1)}{2} \text{ for } 1 \le \mathrm{k} \le n-1$$

$$S(n)=\sum_{i=0}^{n} i = \sum_{i=0}^{n-1} i + n = S(n-1)+n =$$

$$=(n-1)\frac{(n-1+1)}{2}+n=\frac{(n^2-n+2n)}{2}=$$

$$=\frac{n(n+1)}{2}$$

Sum of odd numbers

- **Problem:** Devise a recursive algorithm to add up the first n odd numbers. That is, write a recursive algorithm that returns the following sum on input ***n***

$$\sum_{i=1}^{n}(2i-1) = 1 + 3 + 5 + \ldots + 2n - 1.$$

- **Solution**

$\textsc{Sum-Odd}(n)$

  if $n = 1$ **then**

    return 1

  else

    return $[\textsc{Sum-Odd}(n-1) + (2n-1)]$

Correctness of Sum of odd numbers

- **Claim**: SUM-ODD(n) returns a value equal to $\sum_{i=1}^{n}(2i-1)$ for all natural numbers n.
- **Proof:** by induction on n.
  - **Base Case:** Let $n = 1$. When SUM-ODD(n) is called with $n = 1$, the ***if*** condition is true, thus, SUM-ODD(n) returns 1. Also, $\sum_{i=1}^{1}(2i-1) = 2 - 1 = 1.$
  - **Inductive Hypothesis**: Assume that SUM-ODD(k) returns a value $\sum_{i=1}^{k}(2i-1).$
  - **Inductive Conclusion** (to show): Assume that SUM-ODD(k+1) returns a value equal to $\sum_{i=1}^{k+1}(2i-1).$

Correctness of Sum of odd numbers

- Inductive Step: First note that $k > 1$. Thus, the else case is executed.

- Treating a *return* statement as an assignment we have:

$$\textsc{Sum-Odd}(k+1) = \textsc{Sum-Odd}((k+1)-1) + 2(k+1) - 1$$

$$= \textsc{Sum-Odd}(k) + 2k + 1$$

$$= \sum_{i=1}^{k}(2i-1) + 2k + 1$$

$$= \sum_{i=1}^{k+1}(2i-1)$$

Binary Search

- **Problem:** Determine whether a number $\boldsymbol{x}$ is present in a *sorted* array $\boldsymbol{A}[\boldsymbol{a..b}]$
- **Binary Search Solution:**
  - Compare the middle element $\boldsymbol{mid}$ to $\boldsymbol{x}$
  - If $\boldsymbol{x} = \boldsymbol{mid}$, stop
  - If $\boldsymbol{x} < \boldsymbol{mid}$, throw away larger elements
  - If $\boldsymbol{x} > \boldsymbol{mid}$, throw away smaller elements
  - If there is no element left, $\boldsymbol{x}$ is not in the array

Binary Search Code

```
BinarySearch(A, a, b, x)
If a > b then
    return false
else
    mid ← ⌊(a+b)/2⌋
If x = A[mid] then
    return true
If x < A[mid] then
    return BinarySearch(A, a, mid–1, x)
else
    return BinarySearch(A, mid+1, b, x)
```

Running time calculations: On each iteration, more than half of elements are removed.

Program will run while

$$n^{(0.5)}k > 1$$

$$k < \lg n$$

Correctness of Binary Search

- How do you know if it BinarySearch works correctly?
- First we need to precisely state what the algorithm does through the precondition and postcondition
  - The precondition states what may be assumed to be true initially:
    - Pre: $\boldsymbol{a} \leq \boldsymbol{b} + 1$ and $\boldsymbol{A}[\boldsymbol{a..b}]$ is a sorted array
    - ***found*** = BinarySearch(***A, a, b, x***);
  - The postcondition states what is to be true about the result
    - Post: $\boldsymbol{found} = \boldsymbol{x} \in \boldsymbol{A}[\boldsymbol{a..b}]$ and $\boldsymbol{A}$ is unchanged

Correctness of Recursive Algorithms

- Proof must take us from the precondition to the postcondition.
  - **Base case:** $\boldsymbol{n} = \boldsymbol{b} - \boldsymbol{a} + 1 = 0$
    - The array is empty, so $\boldsymbol{a} = \boldsymbol{b} + \mathbf{1}$
    - The test $\boldsymbol{a} > \boldsymbol{b}$ succeeds and the algorithm correctly returns false
  - **Inductive step:** $\boldsymbol{n} = \boldsymbol{b} - \boldsymbol{a} + 1 > 0$
    - **Inductive hypothesis**:

      Assume BinarySearch($\boldsymbol{A}, \boldsymbol{a'}, \boldsymbol{b'}, \boldsymbol{x}$) returns the correct value for all $\boldsymbol{j}$ such that $0 \le \boldsymbol{j} \le \boldsymbol{n}-1$ where $\boldsymbol{j} = \boldsymbol{b'} - \boldsymbol{a'} + 1$.

Correctness of Recursive Algorithms

- The algorithm first calculates $\boldsymbol{mid} = \lfloor (\boldsymbol{a} + \boldsymbol{b}) / 2 \rfloor$, thus $\boldsymbol{a} \le \boldsymbol{mid} \le \boldsymbol{b}$.
- If $\boldsymbol{x} = \boldsymbol{A}[\boldsymbol{mid}]$, clearly $\boldsymbol{x} \in \boldsymbol{A}[\boldsymbol{a}..\boldsymbol{b}]$ and the algorithm correctly returns true.
- If $\boldsymbol{x} < \boldsymbol{A}[\boldsymbol{mid}]$, since $\boldsymbol{A}$ is sorted (by the precondition), $\boldsymbol{x}$ is in $\boldsymbol{A}[\boldsymbol{a}..\boldsymbol{b}]$ if and only if it is in $\boldsymbol{A}[\boldsymbol{a}..\boldsymbol{mid} - \mathbf{1}]$. By the inductive hypothesis, BinarySearch($\boldsymbol{A}, \boldsymbol{a}, \boldsymbol{mid} - \mathbf{1}, \boldsymbol{x}$) will return the correct value since $0 \le (\boldsymbol{mid} - \mathbf{1}) - \boldsymbol{a} + 1 \le \boldsymbol{n} - \mathbf{1}$.
- The case $\boldsymbol{x} > \boldsymbol{A}[\boldsymbol{mid}]$ is similar

- We have shown that the postcondition holds if the precondition holds and BinarySearch is called.

Summing an Array

- Problem: Given an array of numbers $\boldsymbol{A}[\boldsymbol{a..b}]$ of size $\boldsymbol{n} = \boldsymbol{b} - \boldsymbol{a} + 1 \geq 0$, compute their sum.

```
// Pre: a ≤ b + 1
i ← a, sum ← 0
while i ≠ b + 1 do   // exit condition, called guard G
    sum ← sum + A[i]
    i ← i + 1
```

// Post: $sum = \sum_{j=a}^{b} A[j]$

Correctness of Iterative Algorithms

- The key step in the proof is the invention of a condition called the **loop invariant**, which is supposed to be true at the beginning of an iteration and remains true at the beginning of the next iteration
- The steps required to prove the correctness of an iterative algorithm is as follows:

1. Guess a condition I
2. Prove by induction that I is a loop invariant
3. Prove that $I \wedge \neg G \Rightarrow Postcondition$
4. Prove that the loop is guaranteed to terminate

Correctness of Iterative Algorithms

- In the example, we know that when the algorithm terminates with $i=b+1$, the following condition must hold:

$$sum = \sum_{j=a}^{i-1} A[j]$$

- Use as invariant. Show that at the beginning of the the k-th loop, the condition holds

- **Base Case**: $k = 1$
  - Initialized to $i = a$ and $sum = 0$. Therefore

$$\sum_{j=a}^{i-1} A[j] = 0$$

- **Inductive hypothesis**: Assume at the start of the loop's k-th execution

$$sum = \sum_{j=a}^{i-1} A[j]$$

Correctness of Iterative Algorithms

- Let ***sum'*** and ***i'*** be the values of the variables ***sum*** and ***i*** at the beginning of the $(k+1)$-st iteration.
- In the ***k***-th iteration, the variables were changed as follows:
  - ***sum' = sum + A[i]***
  - ***i' = i + 1***
- Using the inductive hypothesis, we have

$$sum' = sum + A[i] = \sum_{j=a}^{i-1} A[j] + A[i] = \sum_{j=a}^{i} A[j] = \sum_{j=a}^{i'-1} A[j]$$

Correctness of Iterative Algorithms

- We have proven the loop invariant I.
- Now we must show: $I \wedge \neg G \Rightarrow Postcondition$
  - We have $\neg G \Rightarrow i = b + 1$. Substituting into the invariant:

$$sum = \sum_{j=a}^{b+1-1} A[j] = \sum_{j=a}^{b} A[j] \equiv Postcondition$$

- Remains to show that G will eventually be false.
  - Note that i is monotonically increasing since it is incremented inside the loop and not modified elsewhere.
  - From the precondition, i is initialized to $a \leq b+1$.

Summary on Correctness

- How to prove correctness of *recursive algorithm*:
  - Induction
- Proving an algorithm:
  - Precondition
  - Postcondition
- How to prove correctness of *iterative algorithm*
  - Identify a loop invariant condition, and prove it
  - Show that the invariant and terminating condition implies the postcondition
  - Show that the loop is guaranteed to terminate.

Efficiency of algorithms

- Algorithms for solving the same problem can differ dramatically in their efficiency.
- Much more significant than the differences due to hardware and software.
- Comparison of two sorting algorithms (n=10^6 numbers):
  - Insertion sort: $c_1 n^2$
  - Merge sort: $c_2 n$ (lg n)
  - Best programmer (c_1=2), machine language, one billion/second computer.
  - Bad programmer (c_2=50), high-language, ten million/second computer.
  - 2 $(10^6)^2$ instructions/10^9 instructions per second = 2000 seconds.
  - 50 (10^6 lg 10^6) instructions/10^7 instructions per second ≈ 100 seconds.
  - Thus, merge sort on B is 20 times faster than insertion sort on A!
  - If sorting ten million number, 2.3 days VS. 20 minutes.

Asymptotic Efficiency of Recurrences

- Find the asymptotic bounds of recursive equations.
  - Substitution method
  - Recursive tree method
  - Master method (master theorem)
    - Provides bounds for: $T(n) = aT(n/b)+f(n)$ where
      - $a \geq 1$ (the number of subproblems).
      - $b>1$, (n/b is the size of each subproblem).
      - $f(n)$ is a given function.

The Substitution Method

- Two steps:
  - Guess the form of the solution.
    - By experience, and creativity.
    - By some heuristics.
      - If a recurrence is similar to one you have seen before.
        - $T(n)=2T(\lfloor n/2 \rfloor+17)+n$, similar to $T(n)=2T(\lfloor n/2 \rfloor)+n$, , guess $O(n\lg n)$.
      - Prove loose upper and lower bounds on the recurrence and then reduce the range of uncertainty.
        - For $T(n)=2T(\lfloor n/2 \rfloor)+n$, prove lower bound $T(n)= \Omega(n)$, and prove upper bound $T(n)= O(n^2)$, then guess the tight bound is $T(n)=O(n\lg n)$.
    - By recursion tree.
  - Use mathematical induction to find the constants and show that the solution works.

Substitution Method Example

- Solve $T(n)=2T(\lfloor n/2 \rfloor)+n$
- Guess the solution: $T(n)=O(n\lg n)$,
  - i.e., $T(n) \le cn\lg n$ for some c.
- Prove the solution *by induction*:
  - Suppose this bound holds for $\lfloor n/2 \rfloor$, i.e.,
    - $T(\lfloor n/2 \rfloor) \le c\lfloor n/2 \rfloor \lg (\lfloor n/2 \rfloor)$.
  - $T(n) \le 2(c\lfloor n/2 \rfloor \lg (\lfloor n/2 \rfloor))+n$
    - $\le cn \lg (n/2))+n$
    - $= cn \lg n - cn \lg 2 + n$
    - $= cn \lg n - cn + n$
    - $\le cn \lg n$ (as long as $c \ge 1$)

Substitution Method Example

- Boundary (base) Condition
  - In fact, $T(n) = 1$ if $n=1$, i.e., $T(1)=1$.
  - However, $cn \lg n = c \times 1 \times \lg 1 = 0$, which is odd with $T(1)=1$.
- Take advantage of asymptotic notation: it is required $T(n) \le cn \lg n$ hold for $n \ge n_0$, where n_0 is a constant of our choosing.
- Select $n_0 = 2$, thus, $n = 2$ and $n = 3$ as our induction bases. It turns out any $c \ge 2$ suffices for base cases of $n = 2$ and $n = 3$ to hold.

Revise guess

- Guess is correct, but induction proof does not work.
- Problem is that inductive assumption not strong enough.
- Solution: revise the guess by subtracting a lower-order term.
- Example: $T(n)=T(\lfloor n/2 \rfloor)+T(\lceil n/2 \rceil)+1$.
  - Guess $T(n)=O(n)$, i.e., $T(n) \le cn$ for some c.
  - However, $T(n) \le c\lfloor n/2 \rfloor+c\lceil n/2 \rceil+1 = cn+1$, which does not imply $T(n) \le cn$ for any c.
  - Attempting $T(n)=O(n^2)$ will work, but overkill.
  - New guess $T(n) \le cn - b$ will work as long as $b \ge 1$.

Avoid pitfalls

- It is easy to guess $T(n)=O(n)$ (i.e., $T(n) \le cn$) for $T(n)=2T(\lfloor n/2 \rfloor)+n$.
- And wrongly prove:
  - $T(n) \le 2(c\lfloor n/2 \rfloor)+n$
    - $\le cn+n$
    - $=O(n)$. ← wrongly !!!!
- Problem is that it does not prove the <u>*exact form*</u> of $T(n) \le cn$.

Changing Variables

- Suppose $T(n)=2T(\sqrt{n})+\lg n$.
- Rename $m=\lg n$. So $T(2^m)=2T(2^{m/2})+m$.
- Rename $S(m)=T(2^m)$, so $S(m)=2S(m/2)+m$.
  - Which is similar to $T(n)=2T(\lfloor n/2 \rfloor)+n$.
- So the solution is $S(m)=O(m \lg m)$.
- Changing back to $T(n)$ from $S(m)$, the solution is $T(n) = T(2^m) = S(m)= O(m \lg m)=$

$$= O(\lg n \lg \lg n).$$

The Recursion–tree Method (I)

- **Steps:**
  1. Draw the tree based on the recurrence
  2. From the tree determine:
     - # of levels in the tree
     - cost per level
     - # of nodes in the last level
     - cost of the last level (which is based on the number found in 2c)
  3. Write down the summation using $\sum$ notation – this summation sums up the cost of all the levels in the recursion tree

The Recursion–tree Method (II)

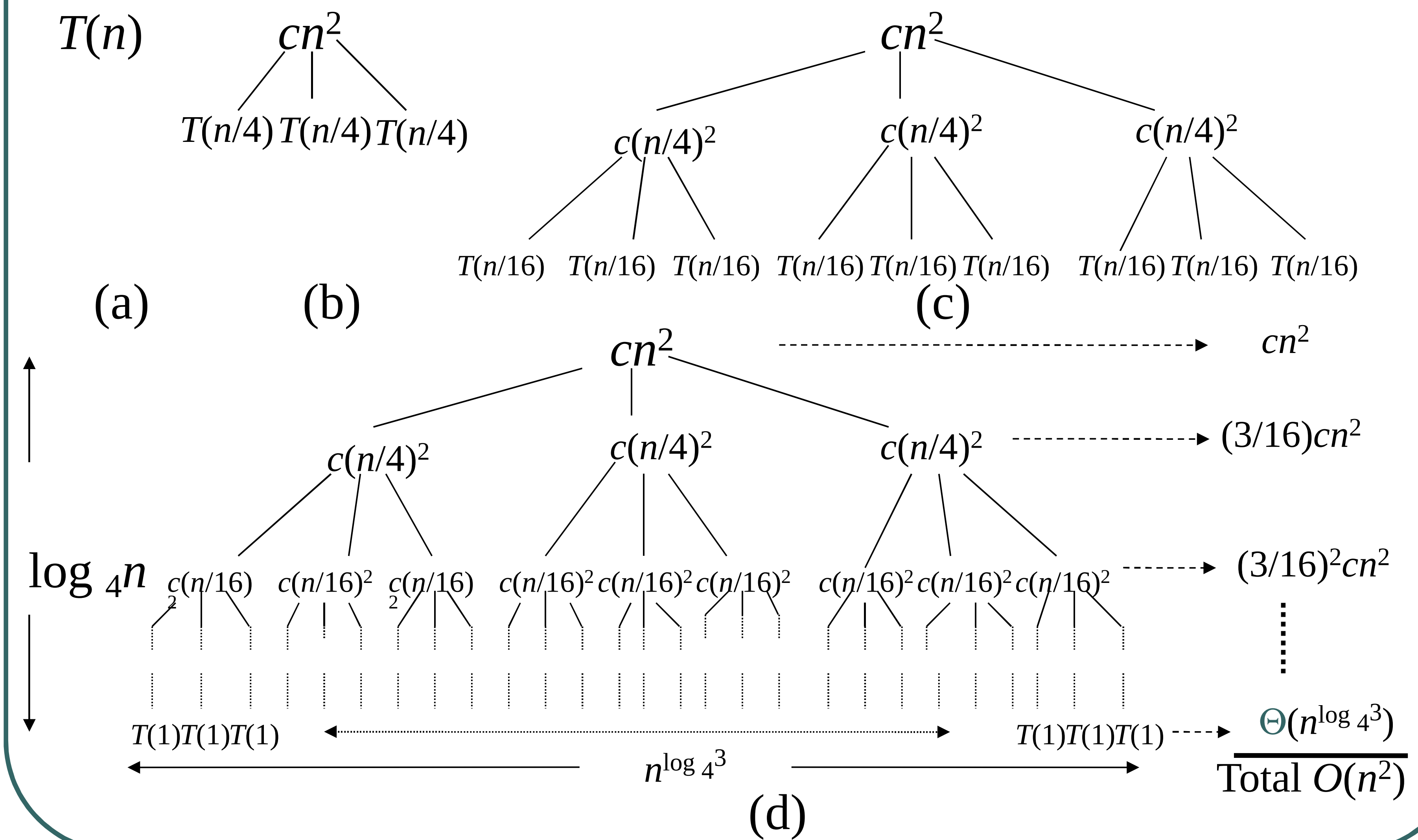
- **Steps:**

4. Recognize the sum or look for a closed form solution for the summation created in 3).
5. Apply that closed form solution to your summation coming up with your “guess” in terms of Big-O, or Θ, or Ω (depending on which type of asymptotic bound is being sought).
6. Then use Substitution Method or Master Method to prove that the bound is correct.

The Recursion–tree Method (III)

- Idea:
  - Each node represents the cost of a single subproblem.
  - Sum up the costs with each level to get level cost.
  - Sum up all the level costs to get total cost.
- Particularly suitable for divide-and-conquer recurrence.
- Best used to generate a good guess, tolerating “sloppiness”.
- If trying to compute cost as exact as possible, then used as direct proof.

Recursion Tree for $T(n)=3T(\lfloor n/4 \rfloor)+\Theta(n^2)$



Solution to $T(n)=3T(\lfloor n/4 \rfloor)+\Theta(n^2)$

- The height is $\log_4 n$,
- #leaf nodes = $3^{\log_4 n} = n^{\log_4 3}$. Leaf node cost: $T(1)$.
- Total cost $T(n)=cn^2+(3/16)\,cn^2+(3/16)^2\,cn^2+$

  $\ldots+(3/16)^{\log_4 (n-1)}\,cn^2+\Theta(n^{\log_4 3})$

  $=(1+3/16+(3/16)^2+\ldots+(3/16)^{\log_4 n-1})\,cn^2+\Theta(n^{\log_4 3})$

  $<(1+3/16+(3/16)^2+\ldots+(3/16)^m+\ldots)\,cn^2+\Theta(n^{\log_4 3})$

  $=(1/(1-3/16))\,cn^2+\Theta(n^{\log_4 3})$

  $=16/13cn^2+\Theta(n^{\log_4 3})$

  $=O(n^2)$.

Prove the previous Guess

- $T(n)=3T(\lfloor n/4 \rfloor)+\Theta(n^2) = O(n^2)$.
- Show $T(n) \le dn^2$ for some d.
- $T(n) \le 3(d(\lfloor n/4 \rfloor)^2) + cn^2$

  $\le 3(d(n/4)^2) + cn^2$

  $= 3/16(dn^2) + cn^2$

  $\le dn^2$, as long as $d \ge (16/13)c$.

Master Method/Theorem

- Used for recurrences of the form

  $T(n) = aT(n/b)+f(n)$, n/b may be $\lceil n/b \rceil$ or $\lfloor n/b \rfloor$.

  where $a \geq 1$, $b > 1$, $f(n)$ be a function.

- Three cases:

  1. If $f(n)=O(n^{\log_b a-\varepsilon})$ for some $\varepsilon > 0$, then $T(n)= \Theta(n^{\log_b a})$.
  2. If $f(n)= \Theta(n^{\log_b a})$, then $T(n)= \Theta(n^{\log_b a} \lg n)$.
  3. If $f(n)=\Omega(n^{\log_b a+\varepsilon})$ for some $\varepsilon > 0$, and if $af(n/b) \leq cf(n)$ for some $c < 1$ and all sufficiently large n, then $T(n)= \Theta(f(n))$.

Implications of Master Theorem

- Comparison between $f(n)$ and $n^{\log_b a}$ (<,=,>)
- Must be asymptotically smaller (or larger) by a polynomial, i.e., n^{ε} for some $\varepsilon>0$.
- In case 3, the "regularity" must be satisfied, i.e., $af(n/b) \leq cf(n)$ for some $c < 1$.
- There are gaps
  - between 1 and 2: $f(n)$ is smaller than $n^{\log_b a}$, but not polynomially smaller.
  - between 2 and 3: $f(n)$ is larger than $n^{\log_b a}$, but not polynomially larger.
  - in case 3, if the "regularity" fails to hold.

Application of Master Theorem

- $T(n) = 9T(n/3)+n;$
  - $a = 9, b = 3, f(n) = n$
  - $n^{\log_b a} = n^{\log_3 9} = \Theta(n^2)$
  - $f(n)=O(n^{\log_3 9-\varepsilon})$ for $\varepsilon=1$
  - By case 1, $T(n) = \Theta(n^2)$.
- $T(n) = T(2n/3)+1$
  - $a=1, b=3/2, f(n) = 1$
  - $n^{\log_b a} = n^{\log_{3/2} 1} = \Theta(n^0) = \Theta(1)$
  - By case 2, $T(n) = \Theta(\lg n)$.

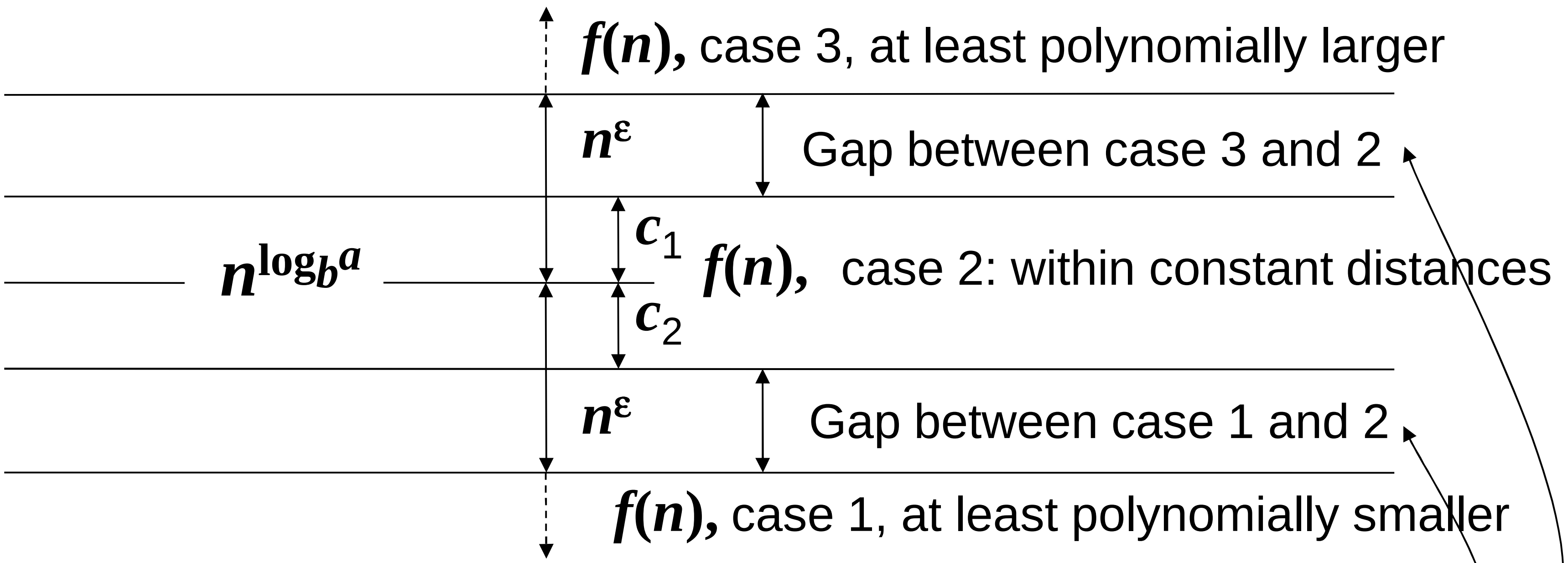
Application of Master Theorem

- $T(n) = 3T(n/4)+n\lg n$;
  - $a=3, b=4$, $f(n) = n\lg n$
  - $n^{\log_b a} = n^{\log_4 3} = \Theta(n^{0.793})$
  - $f(n) = \Omega(n^{\log_4 3+\varepsilon})$ for $\varepsilon \approx 0.2$
  - Moreover, for large n, the "regularity" holds for $c = 3/4$.
    - $af(n/b) = 3(n/4)\lg(n/4) \le (3/4)n\lg n = cf(n)$
  - By case 3, $T(n) = \Theta(f(n)) = \Theta(n\lg n)$.

Exception to Master Theorem

- $T(n) = 2T(n/2)+n \lg n$;
  - a=2, b=2, $f(n) = n\lg n$
  - $n^{\log_b a} = n^{\log_2 2} = \Theta(n)$
  - $f(n)$ is asymptotically larger than $n^{\log_b a}$, but not polynomially larger because
  - $f(n)/n^{\log_b a} = \lg n$, which is asymptotically less than n^{ε} for any $\varepsilon>0$.
  - Therefore, this is a gap between 2 and 3.

Gaps



Notes: 1. for case 3, the regularity also must hold.

2. if $f(n)$, is $\lg n$ smaller, then fall in gap in 1 and 2

3. if $f(n)$ is $\lg n$ larger, then fall in gap in 3 and 2

4. if $f(n) = \Theta(n^{\log_b a} \lg^k n)$, then $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.

Reading

- AHU, chapter 8
- Preiss, chapter: Algorithm Analysis, Asymptotic Notation
- CLR, CLRS, chapters 2, 3, 4
- Notes