Course no. 9

The Theory of Electric Circuits

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About the Course

- **Aims**
  - To provide students with an introduction to electric circuits
- **Objectives**
  - To introduce the concept of resonance in electric circuits

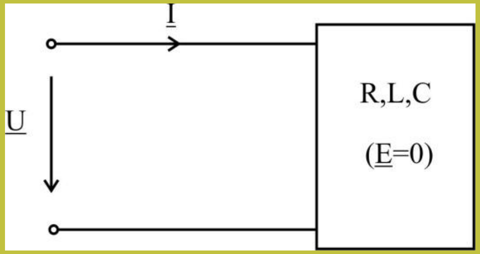
Topics of the course

- **Resonance in electric circuits:**

- **Resonance in a series circuit**
- **Resonance in a parallel circuit**
- **Resonance in real circuits**
- **Resonance in inductively coupled circuits**

Chapter 4

RESONANCE IN ELECTRIC CIRCUITS



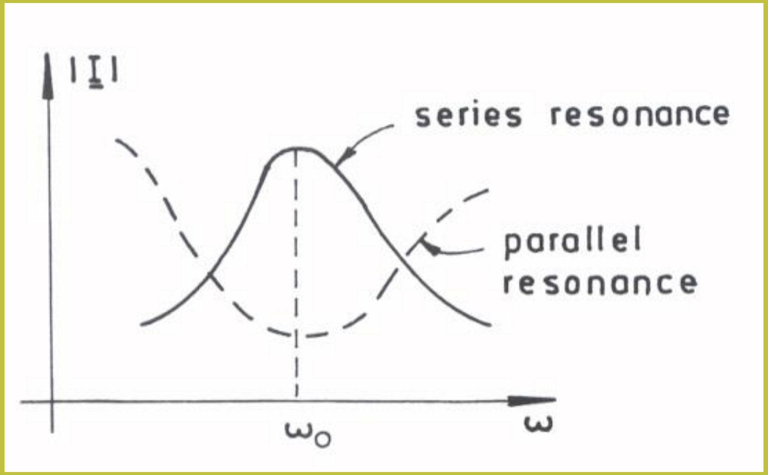
- at resonance : $Q = 0$ (meaning $X = 0$ or $B = 0$)
- the phase shift between I and U is 0 ($\sin \varphi = 0$)

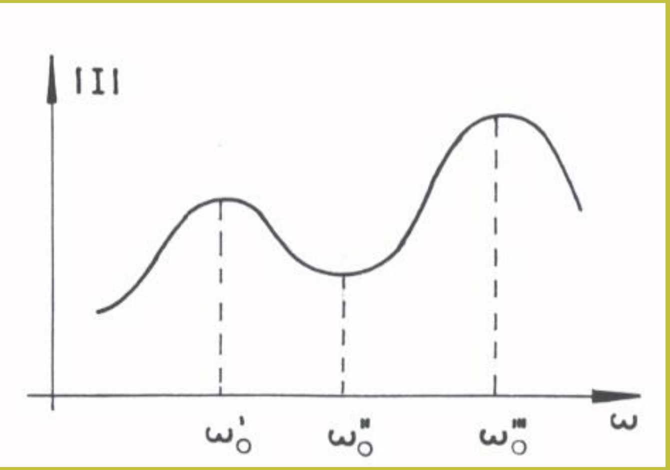
Remarks:

a) ***X* = 0** corresponds to the **series resonance**,

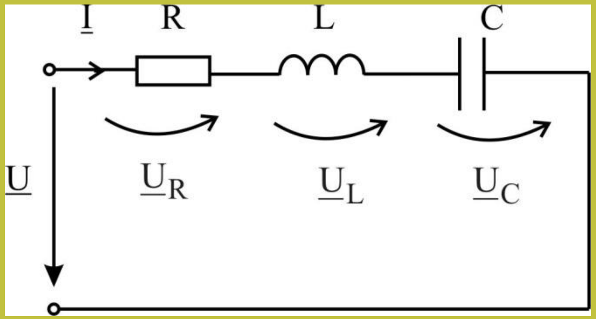
b) ***B* = 0** corresponds to **the parallel resonance**

c) at resonance the current has an **extreme value**





4.1 RESONANCE IN A SERIES CIRCUIT.



$$\underline{Z} = R + j(\omega L - \frac{1}{\omega C})$$

$$I = \frac{U}{Z} = \frac{U}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

-The resonance condition:

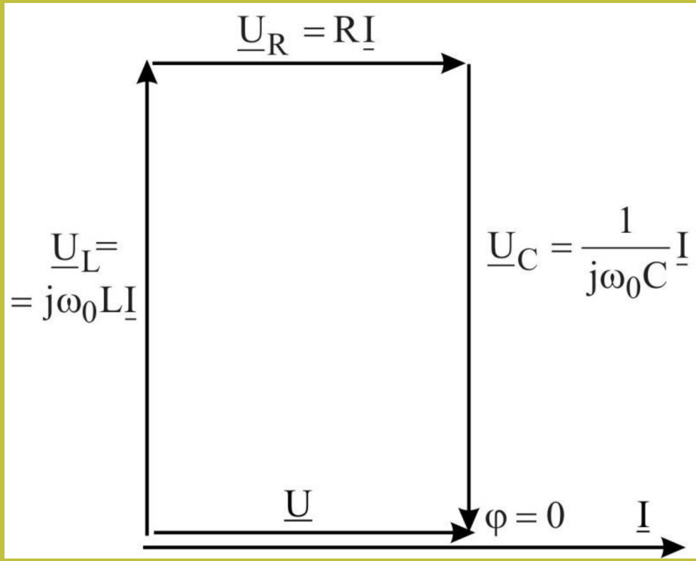
Q = 0=> $X = 0$

$$\varphi = arctg\frac{X}{R} = arctg\frac{\omega L - \frac{1}{\omega C}}{R}$$

$$X = \omega L - \frac{1}{\omega C} = 0$$ or $$\omega L = \frac{1}{\omega C}$$

- *the angular resonant frequency*: $\omega_0 = \frac{1}{\sqrt{LC}}$

The vector diagram (phasorial representation):



Remarks:

a) R does <u>not</u> influence the resonance.

b) $U = U_R = RI$

$Z = \sqrt{R^2 + X^2} = R$ is minimum ($X = 0$),

the **current is maximum**: $I = \frac{U}{R} = I_{max}$

c) $U_L = U_C$: ***voltage resonance***!

d) if $U_L = U_C >> U \rightarrow$ ***overvoltages***.

$$\frac{U_L}{U} = \frac{\omega_0 LI}{RI} = \frac{\omega_0 L}{R} > 1 \text{ or } \omega_0 L > R,$$ because $$\omega_0 = \frac{1}{\sqrt{LC}} \rightarrow \frac{1}{\sqrt{LC}} L = \sqrt{\frac{L}{C}} > R$$

$$\omega_0 L = \frac{1}{\omega_0 C} = \boxed{\sqrt{\frac{L}{C}} = \rho}$$

the characteristic impedance of a series resonant circuit

$$\frac{U_L}{U} = \frac{U_C}{U} = \frac{\rho I}{RI} = \boxed{\frac{\rho}{R} = Q}$$

- the ***quality factor*** (or *Q-factor*)

It shows how many times the voltage across the inductor or across the capacitance of a series resonant circuit (at resonance) is greater than the applied voltage.

$$\boxed{d = \frac{1}{Q} = \frac{R}{\rho}}$$

- the ***damping factor***

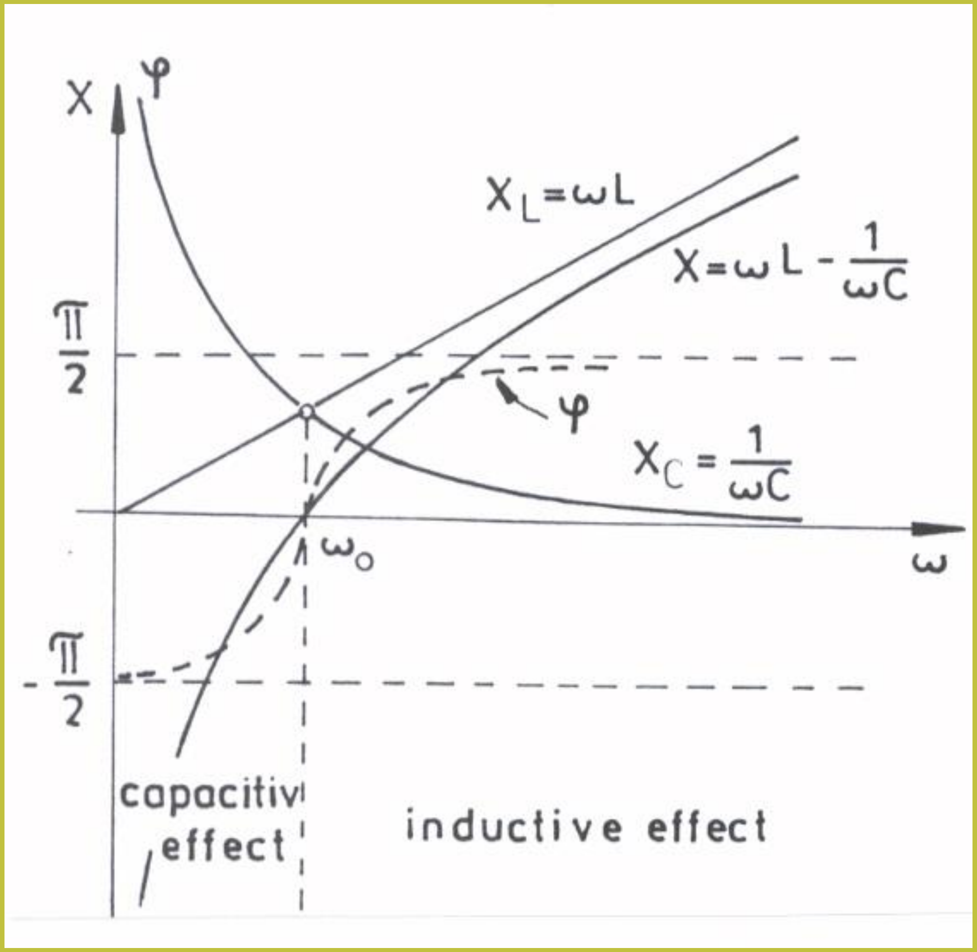
- for the *R*, *L*, *C* series circuit:

$$I = \frac{U}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

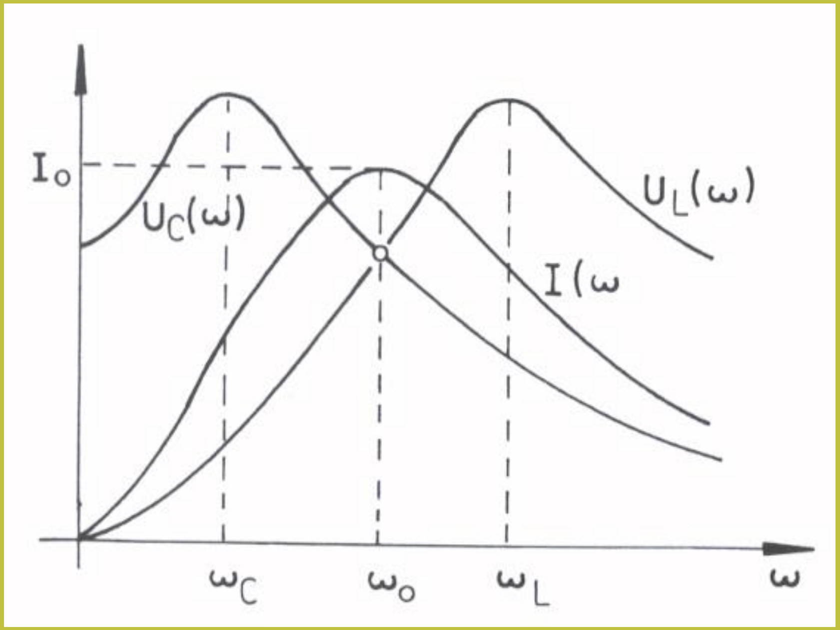
$$X = X_L - X_C = \omega L - \frac{1}{\omega C}$$

$$\varphi = \arctan \frac{\omega L - \frac{1}{\omega C}}{R}$$

Chapter 4. Resonance in electric circuits.



$$U_C = \frac{1}{\omega C} I = \frac{1}{\omega C} \frac{U}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$



$$\frac{\partial U_C}{\partial \omega} = 0 \qquad \boxed{\omega_c = \omega_0 \sqrt{\frac{2 - d^2}{2}}}$$

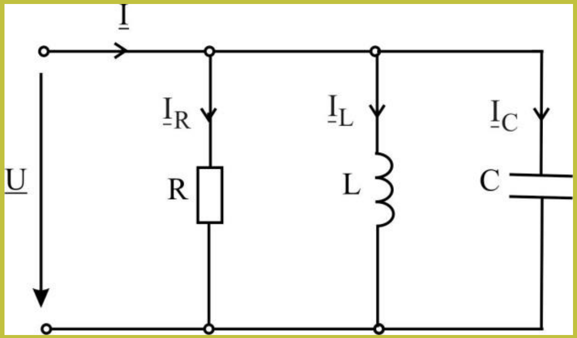
$$U_L = \omega I = \omega L \frac{U}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$\frac{\partial U_L}{\partial \omega} = 0 \qquad \boxed{\omega_L = \omega_0 \sqrt{\frac{2}{2 - d^2}}}$$

Note: $U_{Cmax} = U_{Lmax}$

$$I_0 = \frac{U}{R} \qquad \boxed{\omega_c \le \omega_0 \le \omega_L}$$

4.2 RESONANCE IN A PARALLEL CIRCUIT.



$$\underline{I} = \underline{I}_R + \underline{I}_L + \underline{I}_C$$

$$\underline{I} = \frac{\underline{U}}{R} + \frac{\underline{U}}{j\omega L} + \underline{U}\, j\omega C$$

$$\underline{I} = \underline{U}\left[\frac{1}{R} - j\left(\frac{1}{\omega L} - \omega C\right)\right] \qquad \underline{I} = \underline{U}(G - jB)$$

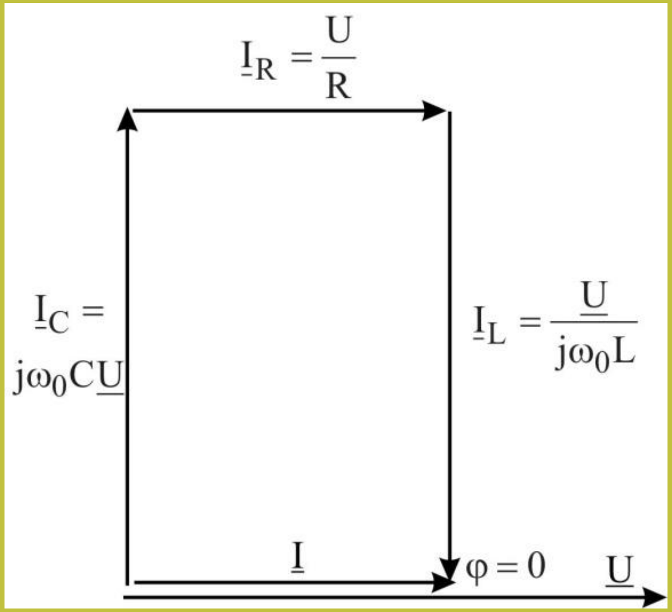
-The resonance condition:

$$Q = 0 \quad => \quad B = 0 \qquad B = \frac{1}{\omega L} - \omega C = 0 \qquad \omega L = \frac{1}{\omega C}$$

- *the angular resonant frequency*: $\omega_0 = \frac{1}{\sqrt{LC}}$

<u>Remark</u>: In practice (that is having a **real** *inductor* and a **real** *capacitor*), the resonant frequencies are **different** for the series and parallel connections.

The vector diagram (phasorial representation):



Remarks:

a) $I = I_R = GU = U/R$

$Y = \sqrt{G^2 + B^2} = G$ is minimum ($B = 0$),

the **current is minimum**: $I = GU = I_{min}$

b) $I_L = I_C$: ***current resonance***!

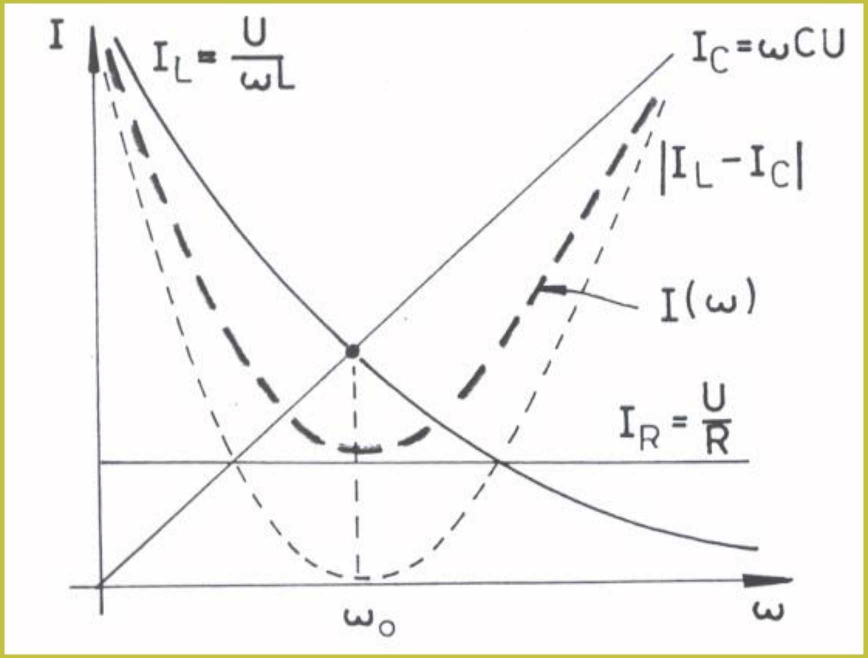
c) if $I_L = I_C >> I$ → ***overcurrents***.

$\omega_0 C = \frac{1}{\omega L} > \frac{1}{R}$ or $\omega_0 L = \frac{1}{\omega C} < R$, because $\omega_0 = \frac{1}{\sqrt{LC}}$

$\omega_0 C = \frac{1}{\omega_0 L} = \sqrt{\frac{C}{L}} = \gamma$ ***the characteristic admittance*** of a parallel resonant circuit

$$Q = \frac{\gamma}{G} = \gamma R$$ - the ***quality factor***

$$d = \frac{1}{Q} = \frac{G}{\gamma}$$ - the ***damping factor***



$$I_R = \frac{U}{R}$$

$$I_L = \frac{U}{\omega L}$$

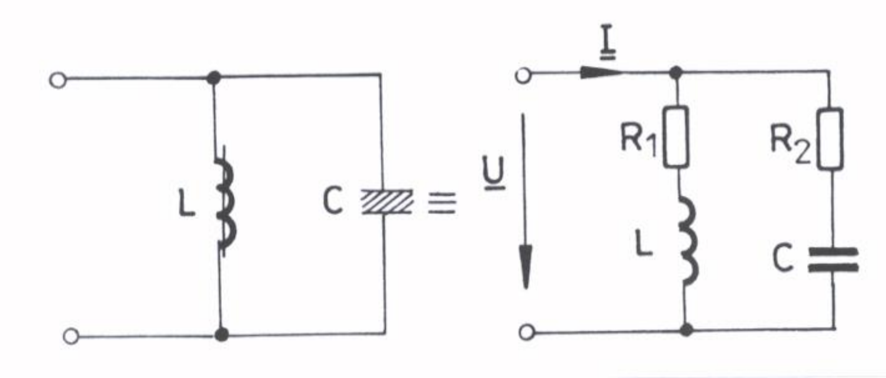
$$I_C = \omega C U$$

$$\underline{I} = \underline{I}_R + \underline{I}_L + \underline{I}_C$$

$$\underline{I} = \underline{U}\left[\frac{1}{R} - j\left(\frac{1}{\omega L} - \omega C\right)\right]$$

$$I = \sqrt{I_R^2 + (I_L - I_C)^2}$$

4.3 RESONANCE IN REAL CIRCUITS.



$$\underline{Z}_1 = R_1 + j\omega L \quad \text{and} \quad \underline{Z}_2 = R_2 + \frac{1}{j\omega C}$$

$$\underline{Y}_1 = \frac{1}{R_1 + j\omega L} \quad \text{and} \quad \underline{Y}_2 = \frac{1}{R_2 - \frac{j}{\omega C}}$$

$$\underline{Y}_e = \underline{Y}_1 + \underline{Y}_2 = \frac{R_1}{R_1^2 + \omega^2 L^2} + \frac{R_2}{R_2^2 + \frac{1}{\omega^2 C^2}} - j\left(\frac{\omega L}{R_1^2 + \omega^2 L^2} - \frac{\frac{1}{\omega C}}{R_2^2 + \frac{1}{\omega^2 C^2}}\right)$$

In a parallel circuit, the resonance occurs when $B_e = 0$

$$B_e = \frac{\omega L}{R_1^2 + \omega^2 L^2} - \frac{\frac{1}{\omega C}}{R_2^2 + \frac{1}{\omega^2 C^2}} = 0$$

$$\omega_0 = \frac{1}{\sqrt{LC}}\sqrt{\frac{\frac{L}{C} - R_1^2}{\frac{L}{C} - R_2^2}} = \frac{1}{\sqrt{LC}}\sqrt{\frac{\rho^2 - R_1^2}{\rho^2 - R_2^2}}$$

Remarks:

a) if $R_1 > \rho$ and $R_2 > \rho$ or $R_1 < \rho$ and $R_2 < \rho$ there is resonance;

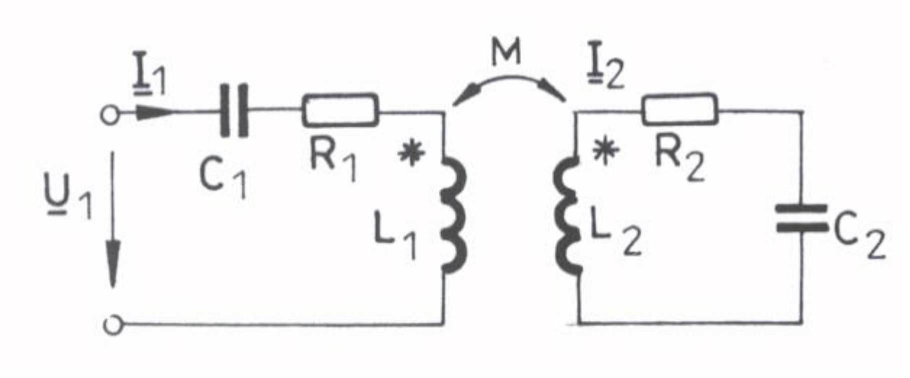
b) if $R_1 < \rho < R_2$ or $R_1 > \rho > R_2$ there is no resonance;

c) if $R_1 = R_2 \neq \rho$, $\omega_0 = \frac{1}{\sqrt{LC}}$ $\rightarrow$ the same as for a series resonant circuit

d) if $R_1 = R_2 = \rho$, $\omega_0 = \frac{0}{0}$ $\rightarrow$ the resonance can occur at any frequency

in this case: $\underline{Z}_e = \frac{\underline{Z}_1 \underline{Z}_2}{\underline{Z}_1 + \underline{Z}_2} = \frac{(R + j\omega L)\left(R - \frac{j}{\omega C}\right)}{2R + j\left(\omega L - \frac{1}{\omega c}\right)} = \rho$

4.4 RESONANCE IN INDUCTIVELY COUPLED CIRCUITS.



$$\begin{cases} \underline{U}_1 = \left[R_1 + j\left(\omega L_1 - \dfrac{1}{\omega C_1} \right) \right] I_1 + j\omega M \underline{I}_2 \\ 0 = \left[R_2 + j\left(\omega L_2 - \dfrac{1}{\omega C_2} \right) \right] \underline{I}_2 + j\omega M \underline{I}_1 \end{cases}$$

It is possible to obtain resonance in the primary circuit, in the secondary circuit or simultaneously in both circuits.

$$\begin{cases} \dfrac{\underline{U}_1}{\underline{I}_1} = \underline{Z}_{e1} = R_1 + jX_1 + j\omega\ M \dfrac{\underline{I}_2}{\underline{I}_1} \\ 0 = \left(R_2 + jX_2 \right) \dfrac{\underline{I}_2}{\underline{I}_1} + j\omega\ M \end{cases} \longrightarrow \quad \dfrac{\underline{I}_2}{\underline{I}_1} = -\dfrac{j\omega M}{R_2 + jX_2}$$

where $X_1 = \omega L_1 - \dfrac{1}{\omega C_1}, X_2 = \omega L_2 - \dfrac{1}{\omega C_2}$

It results:

$$\underline{Z}_{e1} = R_1 + jX_1 + \frac{\omega^2 M^2}{R_2 + jX_2} = R_1 + \frac{R_2 \omega^2 M^2}{R_2^2 + X_2^2} + j\left(X_1 - \frac{\omega^2 M^2 X_2}{R_2^2 + X_2^2} \right)$$

The resonance occurs when $\boldsymbol{X_e = 0}$:

$$X_1 = \frac{\omega^2 M^2 X_2}{R_2^2 + X_2^2}$$

Remarks:

a) the resistance R_1 does not influence the resonance, while R_2 influences the resonance;

b) the realization of the resonance of inductively coupled circuits is important especially in *radio frequencies circuits*, where we can approximate $R_2 \langle\langle X_2$

$$X_1 X_2 = \omega^2 M^2$$

$$\left(\omega L_1 - \frac{1}{\omega C_1}\right)\left(\omega L_2 - \frac{1}{\omega C_2}\right) = \omega^2 M^2$$

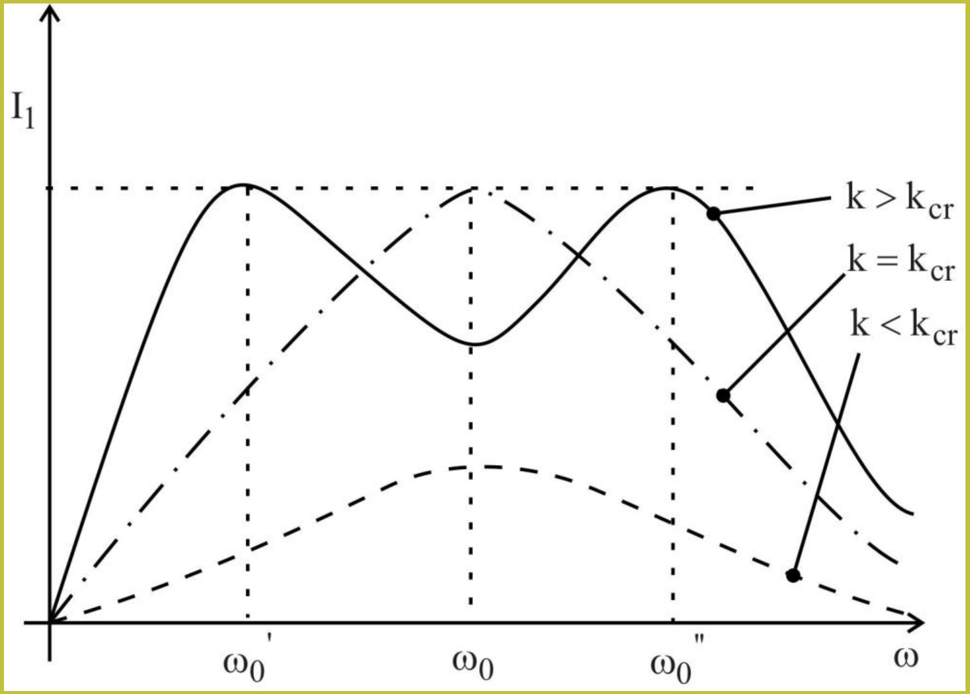
$$\omega^4 (1 - k^2) - \omega^2 (\omega_{01}^2 + \omega_{02}^2) + \omega_{01}\omega_{02} = 0$$

where: $k = \dfrac{M}{\sqrt{L_1 L_2}}$ is the coefficient of coupling,

$\omega_{01} = \dfrac{1}{\sqrt{L_1 C_1}}$ is the resonance frequency of the primary circuit

$\omega_{02} = \dfrac{1}{\sqrt{L_2 C_2}}$ is the resonance frequency of the secondary circuit

$$\omega_0', \omega_0'' = \sqrt{\frac{\omega_{01}^2 + \omega_{02}^2 \pm \sqrt{(\omega_{01}^2 + \omega_{02}^2)^2 - 4(1-k^2)\omega_{01}^2\omega_{02}^2}}{2(1-k^2)}}$$



$$k_{cr} = \sqrt{d^2 - \frac{d^4}{4}} \cong d$$ (critical coupling)