LABORATORY 2

ELECTRICAL MODEL FOR LAPLACE EQUATION IN FINITE DIFFERENCES APPLIED AT EQUIPOTENTIAL SUFACES AND ELECTROSTATIC FIELD SPECTRUM DETERMINATIONS

1. Theoretical facts:

Electrostatic field equations are:

$$div\overline{E} = \frac{\rho_v}{\varepsilon} \text{(Electrical flux low – local form)} \qquad (1)$$

and

$$rot\overline{E} = 0 \text{ (Electrostatic potential theorem, local form)} \qquad (2)$$

The second equation can be writing:

$$\overline{E} = -gradV \qquad (3)$$

Expression which with (1) leads to:

$$divgradV = -\frac{\rho_v}{\varepsilon} \text{ or } \Delta V = -\frac{\rho_v}{\varepsilon} \qquad (4)$$

expression known as Poisson equation.
In Cartesian coordinates it can be write in this form:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon} \qquad (5)$$

In points in which does not exist an electric charge, ρ_v=0, the equation becomes:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad (6)$$

formula known as Laplace equation.
In general the equations (5) and (6) have more than one solution. For a Σ domain, the unique solution can de obtained knowing the conditions on the boundary. This conditions can be: the potential values on the Σ surface, which delimitate the domain (Dirichlet problem), or the field normal component values on the Σ surface (Neumann problem).
Generally the solution cannot be obtained in explicit form. One of the methods for solving the Laplace and Poisson equation is the finite differences method with which the partial differential equations are substituted with algebraic equations named finite differences equations. There exists a form for Laplace equation in finite differences, which will be further established, for the case of plan – parallel field. The Laplace equation is:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad (7)$$

In xO_1y plan we consider a lines network parallel with the O_1x axe, respectively with O_1z placed at the same distance H. Fig.1 (H is called the network step, the cross points are named the nodes, and such a network will be called square network).

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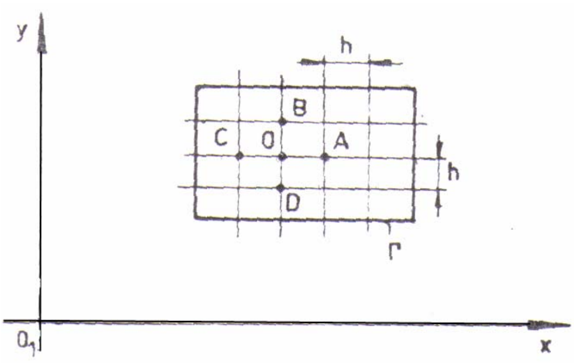

Figure 1

It is considered that in a domain limited by Γ curve is placed a plan-parallel field, with scalar potential V(x, y) which comply with equation (7).
O is a point of x_0, y_0 coordinates and V(x_0, y0) =V_0. The potential V(x, y) is developed in Taylor series around the 0 point till de second order derivation inclusively:

$$V(x,y)=V_0+\frac{1}{1!}\left[(x-x_0)\left(\frac{\partial V}{\partial x}\right)_0+(y-y_0)\left(\frac{\partial V}{\partial y}\right)_0\right]+ \\ +\frac{1}{2!}\left[(x-x_0)^2\left(\frac{\partial^2 V}{\partial x^2}\right)_0+2(x-x_0)(y-y_0)\left(\frac{\partial^2 V}{\partial x\partial y}\right)_0+(y-y_0)^2\left(\frac{\partial^2 V}{\partial y^2}\right)_0\right]+.... \tag{8}$$

The (8) equation is written for A(x_0+h, y0) point:

$$V_A=V_0+h\left(\frac{\partial V}{\partial x}\right)_0+h^2\left(\frac{\partial^2 V}{\partial x^2}\right)_0 \tag{9}$$

Analogous for B, C, D points:

$$V_B=V_0+h\left(\frac{\partial V}{\partial y}\right)_0+h^2\left(\frac{\partial^2 V}{\partial y^2}\right) \tag{10}$$

$$V_C=V_0-h\left(\frac{\partial V}{\partial x}\right)_0+h^2\left(\frac{\partial^2 V}{\partial x^2}\right)_0 \tag{11}$$

$$V_D=V_0-h\left(\frac{\partial V}{\partial y}\right)_0+h^2\left(\frac{\partial^2 V}{\partial y^2}\right)_0 \tag{12}$$

Admitting the 9÷12 equations and taking into account the (7) result:

$$V_0=\frac{V_A+V_B+V_C+V_D}{4} \tag{13}$$

which is the Laplace equation in finite differences corresponding to O(x_0, y0) node.
It can be observed that subtracting (9) and (11) equations, respectively (10) and (12) result:

$$\left(\frac{\partial V}{\partial x}\right)_0=\frac{V_A-V_C}{2h} \tag{14}$$

and $$\left(\frac{\partial V}{\partial y}\right)_0=\frac{V_B-V_D}{2h} \tag{15}$$

i.e. the field in M points is:

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$$\overline{E}_0 = -\left(\frac{\partial V}{\partial x}\right)_0 \bar{i} - \left(\frac{\partial V}{\partial y}\right)_0 \bar{j} = \frac{V_C - V_A}{2h}\bar{i} + \frac{V_D - V_B}{2h}\bar{j} \tag{16}$$

To find the potential V in each branch point inside the Γ boundary (the branch point potentials placed on the boundary are already known –Dirichlet problem) we write an equation like (13) for each of the n internal branch points. It results a system of n equations with n unknown values (branch point potentials) which is generally solved using numerical methods.

Another possibility for checking the potentials which satisfy the equation (13) is to use some analogical methods.

So, consider the model shown in Fig.2, the first Kirchhoff theorem in O point can be written:

$$\frac{V_A - V_0}{R} + \frac{V_B - V_0}{R} + \frac{V_C - V_0}{R} + \frac{V_D + V_0}{R} = 0,$$

or

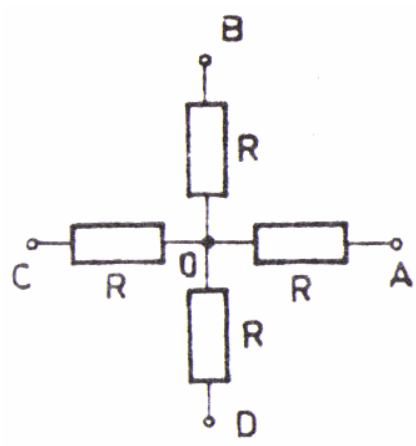

Figure 2

$$V_0 = \frac{V_A + V_B + V_C + V_D}{4}$$

this equation is identical with (13).

Therefore, the Laplace equation in finite differences solutions can be obtained using an adequate electrical circuit arrangement.

2. <u>Work objectives</u>

1. One will write the Laplace equation in finite differences for the system show in Fig.3. One will solve the system (will result $V_1 = V_2 = \frac{V}{8}$; $V_3 = V_4 = \frac{3V}{8}$).

2. One will write the equations adequate with the nodes potential method for the circuit show in Fig.4, the electrical model of the electrostatic regime from Fig.3. One will compare these equations.

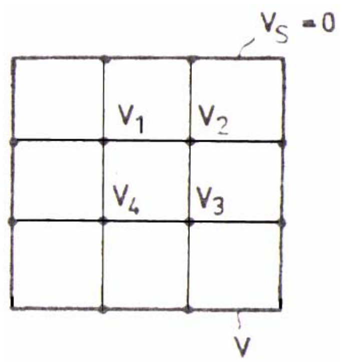

Figure 3

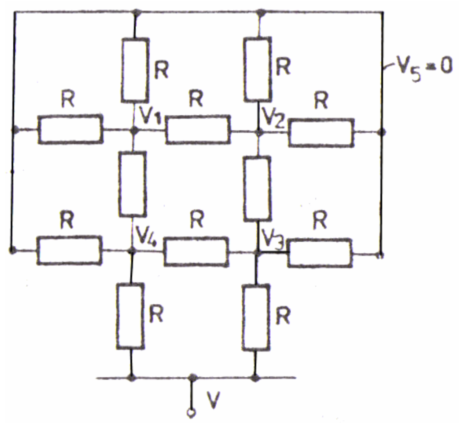

Figure 4

3. One will plot the equipotent curve (surfaces) for the electrostatic regime shown in Fig.5 based on the electrostatic model described in chapter 3.

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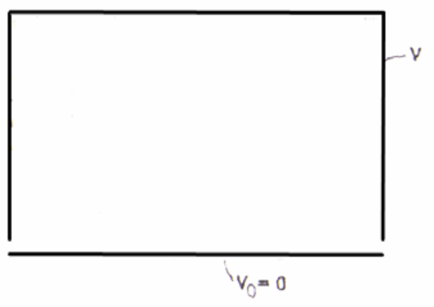

Figure 5

4. One will trace the field lines normal on the equipotent curves.
5. One will determine the electrical field vector (relation (16)) in some nodes, verifying if this is tangent with the curves found at point 4 and normal on the curves determinates at the point 3.

3. <u>The circuit and required equipment</u>

The electrical model corresponding with the system shown in Fig.6 presented in Fig.7. The resistors can be electric bulbs, case in which the equipotent curves can be easily traced.

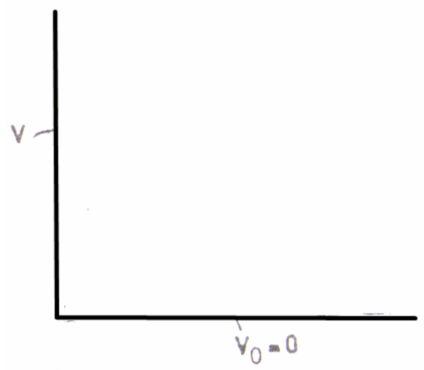

Figure 6

One will realize the circuit arrangement shown in Fig.8, where the symbol signification is:
K – double switch (25 A; 380 V)
AT – autotransformer ATR8; 0-250 V; 8 A;
V_1, V_2 – voltmeter DU+20 on 100 V scale
S – check rod.

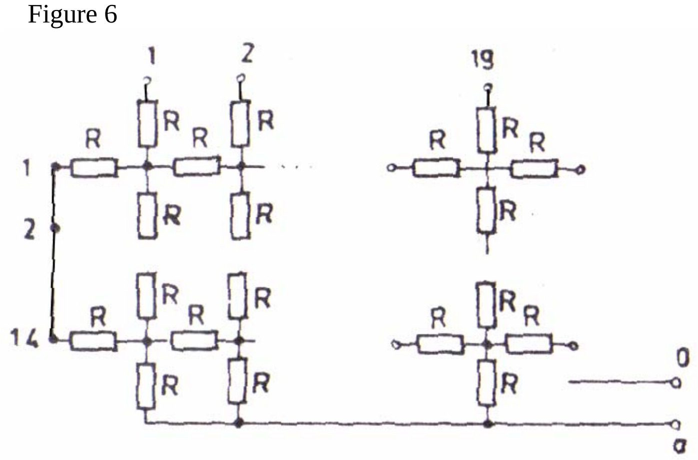

Figure 7

4. <u>Work directions</u>

Turn off the K switch.
Fix the autotransformer AT cursor so as the voltmeter V_1 will indicate 50 V.
One will put the check rod S in each internal network node, reading every time the potential V_{ij}($i=1\div14$, $j=1\div19$). The data measured will be written in a matrix-table 14×19.
The nodes position is fixed on a coordinate paper on which is also drawn the system of the two plates corresponding to the simulated electrostatic regime. For each node the measured potential is written.

5. <u>Experimental result and data processing</u>

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One will answer the questions 1 and 2 from chapter 2.
One will trace the equipotent surfaces on the coordinate paper on which the potential values are written.

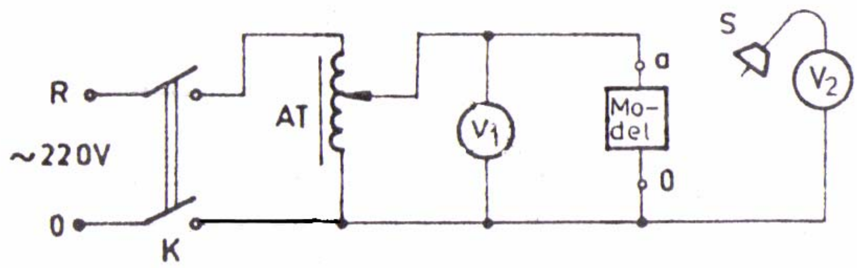

Figure 8

One will trace the field lines such as they will be normal to the equipotent surfaces.
Choose five points and calculate the electrical field vector with formulae (16) and represent them on the same coordinate paper.

6. Observations and conclusions

One will analyze all the error sources that affect the modeling method.
One will determinate the error of this system using the model from Fig. 3, where the exactly result is known.

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