9. Image filtering in the spatial and frequency domains

9.1. Introduction

In this laboratory the convolution operator will be presented. This operator is used in the linear image filtering process applied in the spatial domain (in the image plane by directly manipulating the pixels) or in the frequency domain (applying a Fourier transform, filtering and then applying the inverse Fourier transform. Examples of such filters are: low pass filters (for smoothing) and high pass filters (for edge enhancement).

9.2. The convolution process in the spatial domain

The convolution process implies the usage of a convolution mask/kernel *H* (usually with symmetric shape and size $w \times w$, with *w=2k+1*) which is applied on the source image according to (9.2).

$$I_D(i,j) = H * I_S \tag{9.1}$$

$$I_D(i,j) = \sum_{u=0}^{w-1} \sum_{v=0}^{w-1} H(u,v) \cdot I_S(i+u-k, j+v-k) \tag{9.2}$$

This implies the scanning of the source image I_S, pixel by pixel, ***ignoring the first and last k rows and columns*** (Fig. 9.1) and the computation of the intensity value in the current position *(i, j)* of the destination image I_D using (9.2). The convolution mask is positioned spatially with its central element over the current position *(i, j)*.

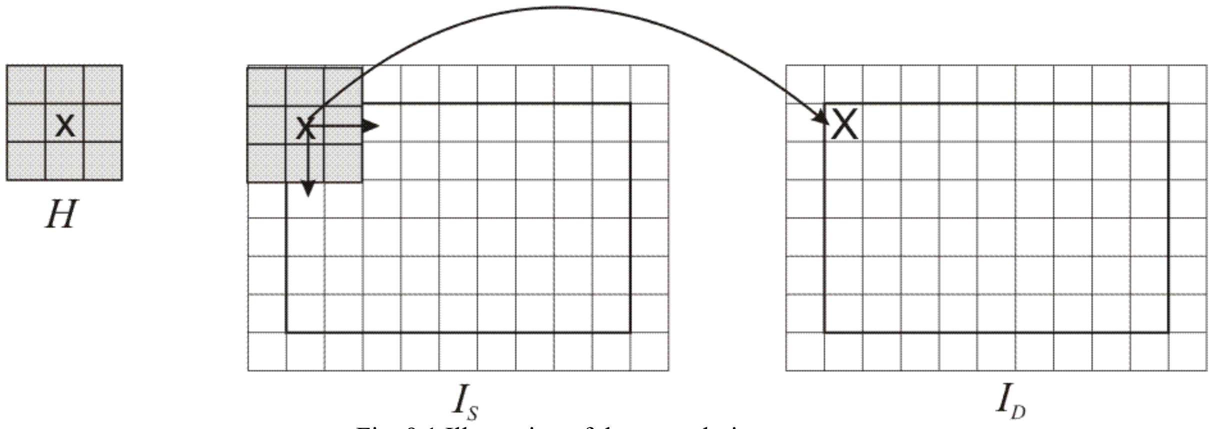


Fig. 9.1 Illustration of the convolution process.

The convolution kernels can have also non-symmetrical shapes (the central/reference element is not positioned in the center of symmetry). Convolution with such kernels is applied in a similar way, but such examples will not be presented in the current laboratory.

9.2.1. Low-pass filters

Low-pass filters are used for image smoothing and noise reduction (see the lecture material). Their effect is an averaging of the current pixel with the values of its neighbors, observable as a “blurring” of the output image (they allow to pass only the low frequencies of the image).

All elements of the kernels used for low-pass filtering have positive values. Therefore, a common practice used to scale the result in the intensity domain of the output image is to divide the result of the convolution with the sum of the elements of the kernel:

$$I_D(i,j) = \frac{1}{c}\sum_{u=0}^{w-1}\sum_{v=0}^{w-1} H(u,v)\cdot I_S(i+u-k, j+v-k) \tag{9.3}$$

where:

$$c = \sum_{u=0}^{w-1}\sum_{v=0}^{w-1} H(u,v) \tag{9.4}$$

Example kernel matrices:

Mean filter (3x3):

$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{9.5}$$

Gaussian filter (3x3):

$$\frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \tag{9.6}$$

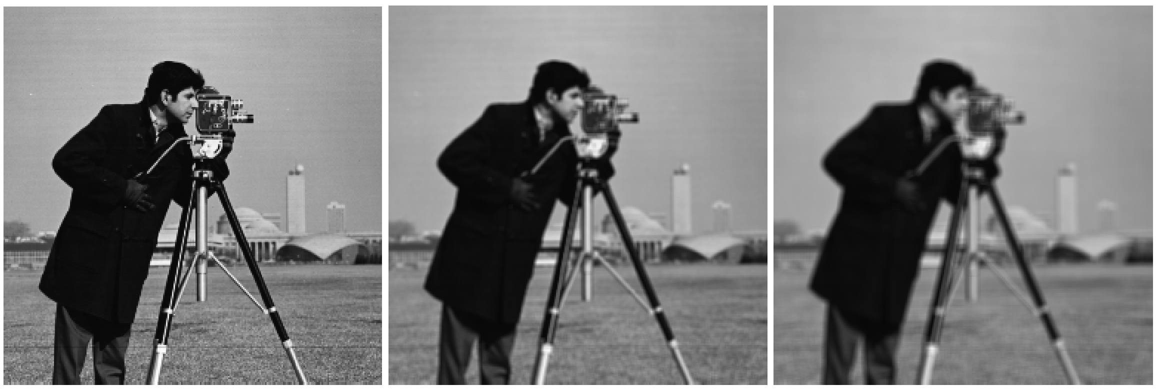


Fig. 9.2 a. Original image; b. Result obtained by applying a 3x3 mean filter. c. Result obtained by applying a 5x5 mean filter.

9.2.2. High-pass filters

These filters will highlight regions with step intensity variations, such as edges (will allow to pass the high frequencies).

The kernels used for edge detection have the sum of their elements equal to 0:

Laplace filters (edge detection) (3x3):

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix} \tag{9.7}$$

or

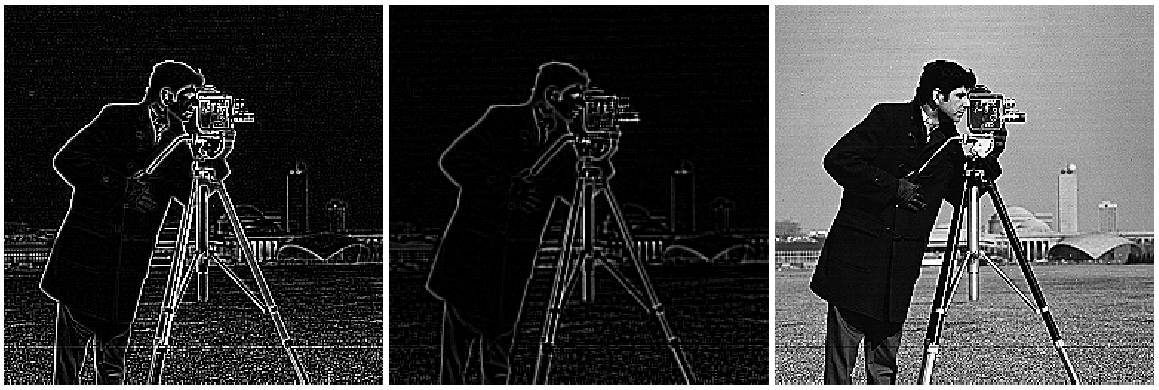
$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix} \tag{9.8}$$

High-pass filters (3x3):

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix} \tag{9.9}$$

or

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 9 & -1 \\ -1 & -1 & -1 \end{bmatrix} \tag{9.10}$$



a. b. c.

Fig. 9.3 a. The result of applying the Laplace edge detection filter (9.8) on the original image (Fig. 9.2a); b. The result of applying the Laplace edge detection filter (9.8) on the blurred image from Fig. 9.2b (previously filtered with the 3x3 mean filter); c. The result obtained by filtering the original image with the high-pass filter (9.10)

9.3. Image filtering in the frequency domain

The 1D discrete Fourier transform (DFT) of an array of N real or complex numbers is an array of N complex numbers, given by:

$$X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi jkn}{N}}, \quad k = \overline{0...N-1} \tag{9.11}$$

The inverse discrete Fourier transform (IDFT) is given by:

$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{\frac{2\pi jkn}{N}}, \quad n = \overline{0...N-1} \tag{9.12}$$

The 2D DFT is performed by applying the 1D DFT on each row of the input image and then on each column of the previous result. The 2D IDTF is performed by applying the 1D IDFT on each column of the DFT “image” and then on each row of the previous result. The set of complex numbers which are the result of the DFT may also be represented in polar coordinates (magnitude, phase). The set of (real) magnitudes represent the frequency power spectrum of the original array.

The DFT and its inverse are usually performed using the Fast Fourier Transform recursive approach, which reduces the computation time from $O(n^2)$ to $O(n \ln n)$, which represents a significant speed increase, especially in the case of 2D image processing, where a $O(n^2m^2)$ complexity would be intractable for large images as opposed to the almost linear in number of pixels $O(nm\ln(nm))$ complexity.

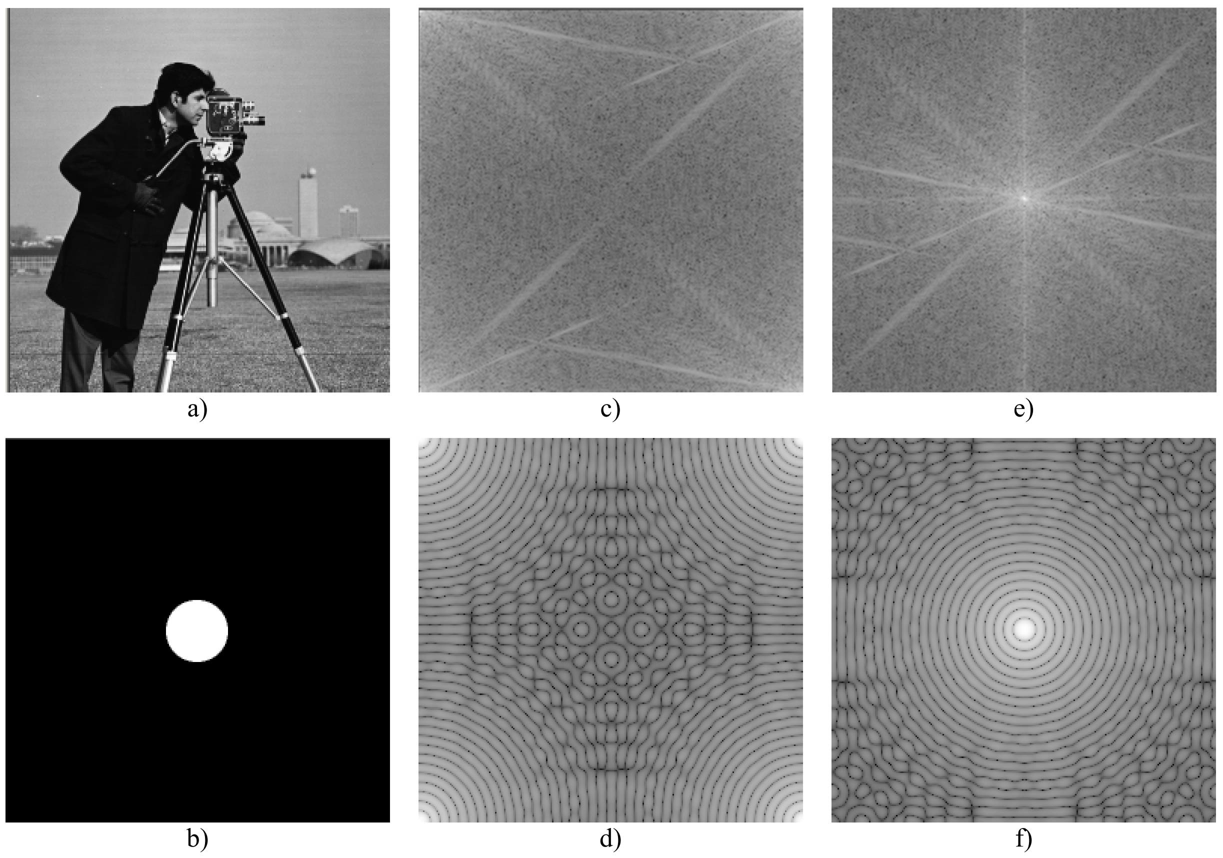


Fig. 9.4 a) and b) original images; c) and d) logarithm of magnitude spectra; e) and f) centered logarithm of magnitude spectra

9.3.1. Aliasing

The aliasing phenomenon is a consequence of the Nyquist frequency limit (a sampled signal cannot represent frequencies higher than half the sampling frequency). This means that the higher half of the frequency domain representation is redundant. This fact can also be seen from the identity:

$$X_k = X_{N-k}^* \tag{9.13}$$

(where the asterisk denotes complex conjugation) which is true if the input numbers x_k are real. Therefore, the typical 1D Fourier spectrum will contain the low frequency components in both the lower and upper part, with high frequency located symmetrically about the middle. In 2D, the low frequency components will be located near the image corners and the high frequency components in the middle (see Fig. 9.4c, d). This makes the spectrum hard to read and interpret. In order to center the low frequency components spectrum about the middle of the spectrum, one should first perform the transformation on the input data:

$$x_k \leftarrow (-1)^k x_k \tag{9.14}$$

In 2D the centering transformation becomes:

$$x_{uv} \leftarrow (-1)^{u+v} x_{uv} \tag{9.15}$$

After applying this centering transform, in 1D the spectrum will contain the low frequency components in the center, and the high frequency components will be located symmetrically toward the left and right ends of the spectrum. In 2D, the low frequency components will be located in the middle of the image, while various high frequency components will be located toward the edges.

The magnitudes located on any line passing through the DFT image center represent the 1D frequency spectrum components of the original image, along the direction of the line. Every such line is therefore symmetrical about its middle (the image center).

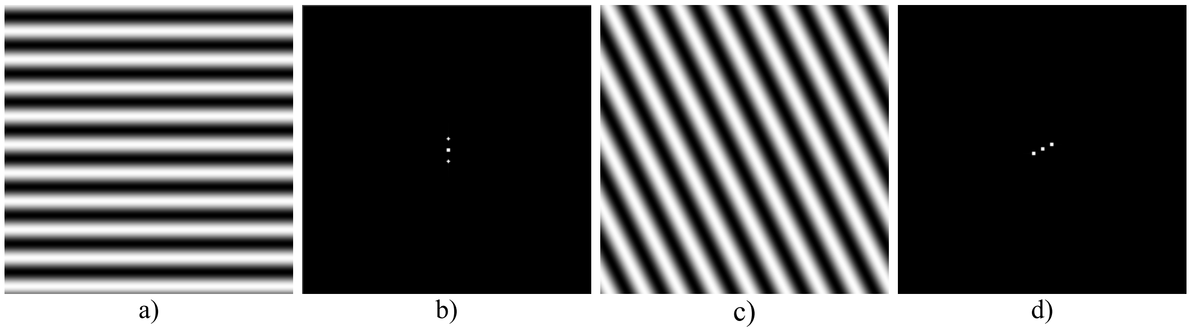


Fig. 9.5 Fourier transforms of sine image waves a) and c). The center point in b) and d) represent the DC component, the other two symmetrical points are due to the sine wave frequency.

9.3.2. Ideal low-pass and high-pass filters in frequency domain

The convolution in spatial domain is equivalent to scalar multiplication in frequency domain. Therefore, especially for large convolution kernels, it is computationally convenient to perform convolution in the frequency domain.

The algorithm for filtering in the frequency domain is:

a) Perform the image centering transform on the original image (9.15)
b) Perform the DFT transform
c) Alter the Fourier coefficients according to the required filtering
d) Perform the IDFT transform
e) Perform the image centering transform again (this undoes the first centering transform).

An ideal low pass filter will alter all the Fourier coefficients that are further away from the image center *(W/2, H/2)* than a given distance *R*, by turning them to zero (*W* is the image width and *H* is the image height):

$$X'_{uv} = \begin{cases} X_{uv}\ , & \left(\frac{H}{2}-u\right)^2 + \left(\frac{W}{2}-v\right)^2 \le R^2 \\ 0\ , & \left(\frac{H}{2}-u\right)^2 + \left(\frac{W}{2}-v\right)^2 > R^2 \end{cases} \tag{9.16}$$

An ideal high-pass filter will alter all Fourier coefficients that are at a distance less than *R* from the image center *(W/2, H/2)*, by turning them to 0.

$$X'_{uv} = \begin{cases} X_{uv}\ , & \left(\frac{H}{2}-u\right)^2 + \left(\frac{W}{2}-v\right)^2 > R^2 \\ 0\ , & \left(\frac{H}{2}-u\right)^2 + \left(\frac{W}{2}-v\right)^2 \le R^2 \end{cases} \tag{9.17}$$

The results of filtering with ideal low- and high-pass filtering are presented in Fig. 9.6 b) and c). Unfortunately, the corresponding spatial filters Fig. 9.6 e) and d) are not FIR (they have an infinite support) and keep oscillating away from their centers. Because of this, the low-pass and high-pass filtered images have a disturbing ringing wavy aspect. In order to correct this, the cutoff in the frequency domain must be smoother, as presented in the next section.

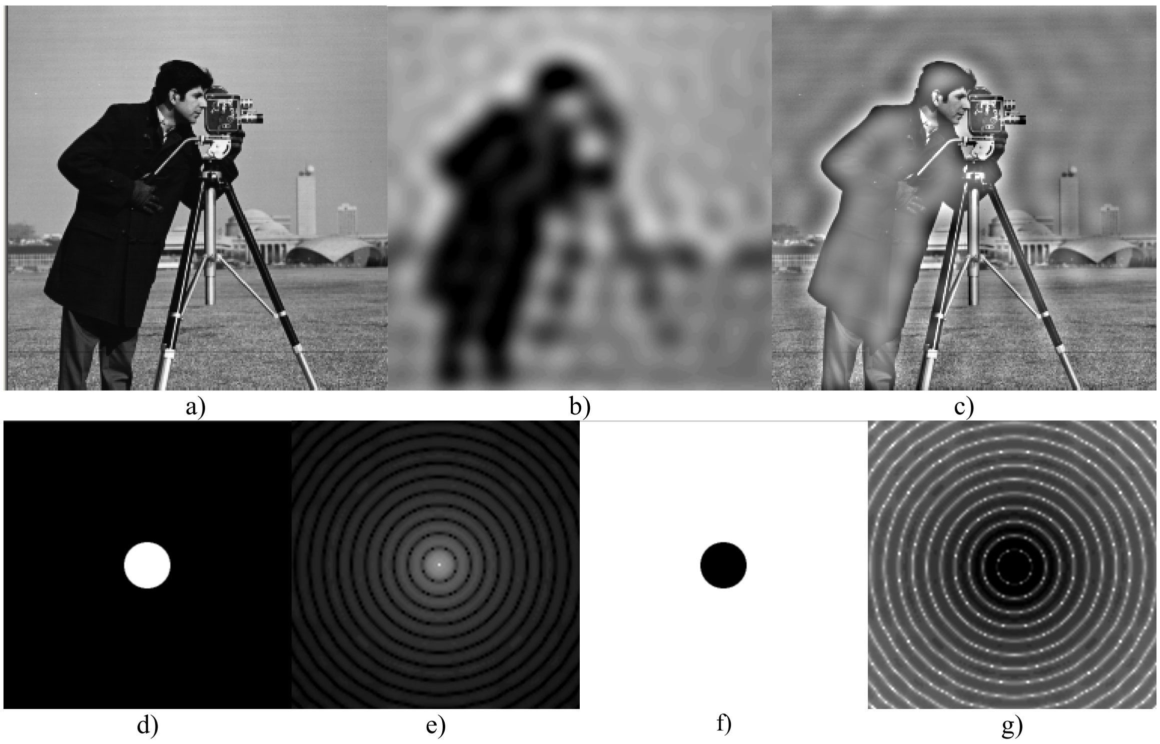


Fig. 9.6 a) original image; b) result of ideal low-pass filtering; c) result of ideal high-pass filtering; d) ideal low-pass filter in the frequency domain; e) corresponding ideal low-pass filter in the spatial domain; f) ideal high-pass filter in the frequency domain; g) corresponding ideal high-pass filter in the spatial domain

9.3.3. Gaussian low-pass and high-pass filtering in the frequency domain

In the case of Gaussian filtering, the frequency coefficients are not cut abruptly, but smoother cutoff process is used instead. This also takes advantage of the fact that the DFT of a Gaussian function is also a Gaussian function (Fig. 9.7d-g).
The Gaussian low-pass filter attenuates frequency components that are further away from the image center *(W/2, H/2)*. $A \sim \frac{1}{\sigma}$ where σ is the standard deviation of the equivalent spatial domain Gaussian filter.

$$X'_{uv} = X_{uv} e^{-\frac{\left(\frac{H}{2}-u\right)^2+\left(\frac{W}{2}-v\right)^2}{A^2}} \tag{9.18}$$

The Gaussian high-pass filter attenuates frequency components that are near to the image center *(W/2, H/2)*:

$$X'_{uv} = X_{uv} \left(1 - e^{-\frac{\left(\frac{H}{2}-u\right)^2+\left(\frac{W}{2}-v\right)^2}{A^2}} \right) \tag{9.19}$$

Fig. 9.7 shows the results of Gaussian filter. Notice that the ringing (wavy) effect visible in Fig. 9.6 disappeared.

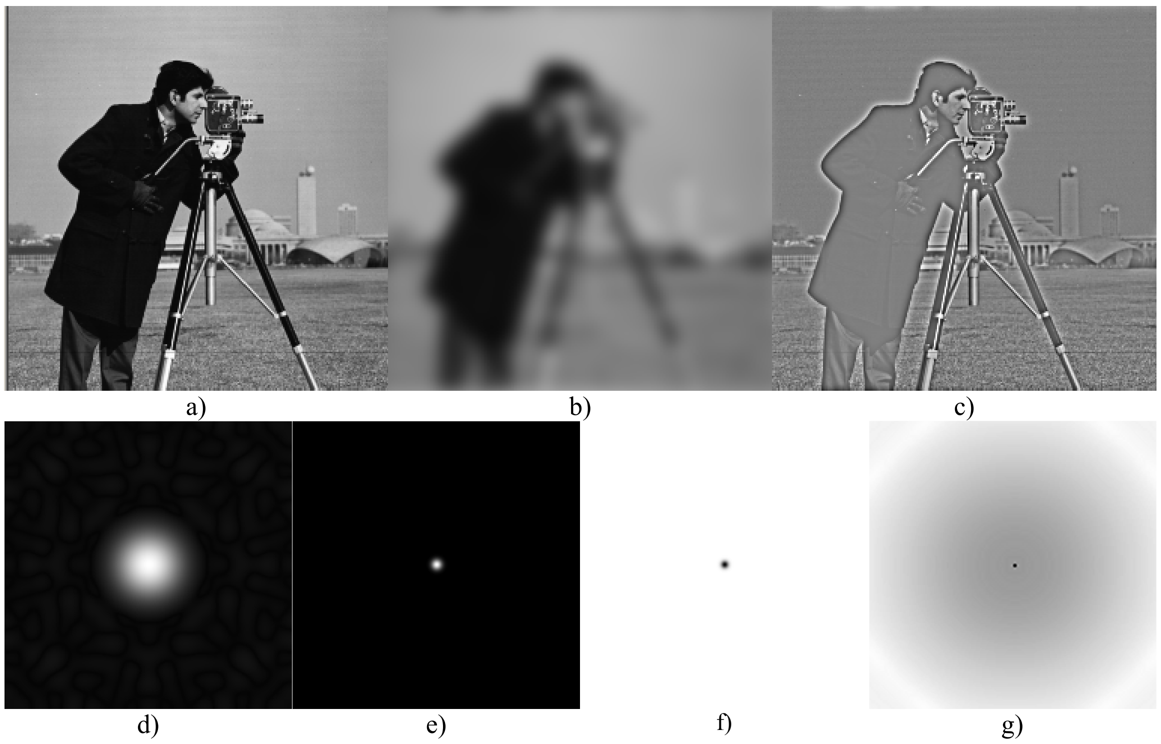


Fig. 9.7 a) original image; b) result of Gaussian low-pass filtering; c) result of Gaussian high-pass Filtering; d) Gaussian low-pass filter in the frequency domain; e) corresponding Gaussian low-pass filter in the spatial domain; f) Gaussian high-pass filter in the frequency domain; g) corresponding Gaussian high-pass filter in the spatial domain

9.4. Implementation details

9.4.1. Spatial domain filters

Low-pass filters will always have positive coefficients, and therefore, the resulting filtered image will have positive values. You must ensure that the resulting image fits in the desired range (0-255 in our case). In order to ensure this, you must ensure that the coefficients of a low-pass filter sum to 1. If you are using integer operations pay attention to the order of operations! Usually, the division should be the last operation performed in order to minimize the rounding errors!

High-pass filters will have both positive and negative coefficients. You must ensure that the final result is an integer between 0 and 255! There are three possibilities to ensure that the resulting image fits the destination range. The first one is to compute:

$$
\begin{gathered}
S_{+} = \sum_{F_k > 0} F_k, \quad S_{-} = \sum_{F_k < 0} -F_k, \\
S = \frac{1}{2\max\{S_{+}, S_{-}\}} \\
I_D(u,v) = S(F * I_S)(u,v) + \left\lfloor \frac{L}{2} \right\rfloor
\end{gathered}
\tag{9.20}
$$

In the formula above S_+ represents the sum of positive filter coefficients and S_- the sum of negative filter coefficients magnitudes. This result of applying the high-pass filter always lies in the interval $[-LS_-, LS_+]$ where L is the maximum image gray level (255). The result of this transform will place scale the result to [-*L/2, L/2*] and then move the 0 level to *L/2*.

Another approach is to perform all operations using signed integers determine the minimum and maximum and then linearly transform the resulting values according to:

$$D = \frac{L(S - \min)}{\max - \min} \tag{9.21}$$

The third approach is to compute the magnitude of the result and saturate everything that exceeds the maximum level *L*.

9.4.2. Frequency domain filters

It is common practice for visualization and for processing purposes to consider a representation of the frequency space which has the (0,0) coefficient in the image center. This can be achieved by cross-swapping the four quadrants of the Fourier image channels. Equivalently, we can preprocess the source image using 9.15. The generic filter presented below uses the following helper function which performs the centering operation.

```
void centering_transform(Mat img){
	//expects floating point image
	for (int i = 0; i < img.rows; i++){
		for (int j = 0; j < img.cols; j++){
			img.at<float>(i, j) = ((i + j) & 1) ? -img.at<float>(i, j) : img.at<float>(i, j);
		}
	}
}
```

The OpenCV library provides an implementation for performing Discrete Fourier Transform. The following template code performs both the direct and the inverse transformation. Processing should be done on the magnitude channel of the Fourier transform. Since DFT works best if the input image has dimensions equal to powers of two, use *cameraman.bmp* as your input.

```
Mat generic_frequency_domain_filter(Mat src){
	//convert input image to float image
	Mat srcf;
	src.convertTo(srcf, CV_32FC1);

	//centering transformation
	centering_transform(srcf);

	//perform forward transform with complex image output
	Mat fourier;
	dft(srcf, fourier, DFT_COMPLEX_OUTPUT);

	//split into real and imaginary channels
	Mat channels[] = { Mat::zeros(src.size(), CV_32F), Mat::zeros(src.size(), CV_32F) };
	split(fourier, channels);  // channels[0] = Re(DFT(I)), channels[1] = Im(DFT(I))

	//calculate magnitude and phase in floating point images mag and phi
	Mat mag, phi;
	magnitude(channels[0], channels[1], mag);
	phase(channels[0], channels[1], phi);

	//display the phase and magnitude images here
	// ......
	//insert filtering operations on Fourier coefficients here
	// ......

	//store in real part in channels[0] and imaginary part in channels[1]
	// ......

	//perform inverse transform and put results in dstf
	Mat dst, dstf;
	merge(channels, 2, fourier);
	dft(fourier, dstf, DFT_INVERSE | DFT_REAL_OUTPUT | DFT_SCALE);

	//inverse centering transformation
	centering_transform(dstf);

	//normalize the result and put in the destination image
	normalize(dstf, dst, 0, 255,  NORM_MINMAX, CV_8UC1);

	return dst;
}
```

9.5. Practical work

1. Implement a general filter which performs the convolution operator with a custom kernel matrix. The scaling coefficient should be computed automatically as either the reciprocal of the sum of filter coefficients for low pass filters or according to equation (9.20) for high-pass filters.
2. Test the filter with the kernels from equations (9.5) .... (9.10)
3. Study the provided generic function for processing in the frequency domain. Perform the conversion of a source image from spatial domain to frequency domain by using the Fourier transform (DFT), then apply the inverse Fourier transform (IDFT) on the obtained Fourier spectrum coefficients and check if the destination is the same as the source image.
4. Add a processing function that computes and displays the logarithm of the magnitude of the Fourier transform of an input image. Add 1 to the magnitude to avoid *log*(0).
5. Add processing functions that perform low- and high-pass filtering in the frequency domain using the ideal and Gaussian filters from equations (9.16)...(9.19).
6. **Save your work. Use the same application in the next laboratories. At the end of the image processing laboratory you should present your own application with the implemented algorithms.**

References

[1]. Umbaugh Scot E, *Computer Vision and Image Processing*, Prentice Hall, NJ, 1998, ISBN 0-13-264599-8

[2] R.C.Gonzales, R.E.Woods, *Digital Image Processing, 2-nd Edition,* Prentice Hall, 2002