Rezolvare subiecte Analiză matematică la Inginerie electrică restanțe toamnă 2025

Note Title 9/13/2025

I

1. a) Funcția exponențială e^z, $z \in \mathbb{C}$.

   b) $\operatorname{Im}(w) = ?$ $\quad w = e^{-\frac{1}{2} + \frac{3\pi}{4} i}$.

2. a) Seria Taylor pentru funcția sinus

   b) Seria Taylor pentru $f(x) = x\sin(x^2)$, $x_0 = 0$.

3. a) Divergența unui câmp vectorial (definiție + 3 proprietăți)

   b) $\operatorname{div} \vec{v} = ?$ $\quad \vec{v} = (3x^2+y)\,\vec{i} - yz^3\,\vec{j} + z\cdot e^{2z-x}\,\vec{k}$

4. a) Funcția Gamma (definiție + 3 proprietăți)

   b) $\int_0^\infty x^2 \cdot e^{-x^2}\,dx = ?$

1. a) $e^{z} = e^{a+bi} = e^{a}(\cos b + i \sin b)$.

b) $w = e^{-\frac{1}{2} + \frac{3\pi i}{4}} = e^{-\frac{1}{2}}\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right) = \frac{1}{\sqrt{e}}\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$

$$\cos \frac{3\pi}{4} = \cos \frac{4\pi - \pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = \underbrace{\cos \pi}_{-1} \underbrace{\cos \frac{\pi}{4}}_{\frac{\sqrt{2}}{2}} + \underbrace{\sin \pi}_{0} \cdot \underbrace{\sin \frac{\pi}{4}}_{\frac{\sqrt{2}}{2}} = -\frac{\sqrt{2}}{2}$$

$$\sin \frac{3\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right) = \underbrace{\sin \pi}_{0} \cdot \underbrace{\cos \frac{\pi}{4}}_{\frac{\sqrt{2}}{2}} - \underbrace{\cos \pi}_{-1} \cdot \underbrace{\sin \frac{\pi}{4}}_{\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \quad w = \frac{1}{\sqrt{e}}\left(-\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2\sqrt{e}} + i\,\frac{\sqrt{2}}{2\sqrt{e}}$$

$\Rightarrow \quad \operatorname{Im} w = \frac{\sqrt{2}}{2\sqrt{e}}$. partea imaginară a lui w.

2. a) $$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad x \in \mathbb{C}.$$

b) $$f(x) = x \sin(x^2) = x \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!}$$

$$= x \cdot \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)!}, \quad x \in \mathbb{C}.$$

$$= x^3 - \frac{x^7}{6} + \dots$$

3. a) $\vec{v} = P\vec{i} + Q\vec{j} + R\vec{k}$ $\quad \operatorname{div} \vec{v} = P'_x + Q'_y + R'_z$

$$\operatorname{div}(c\cdot\vec{v}) = c\cdot\operatorname{div}(\vec{v})$$

$$\operatorname{div}(\vec{v}+\vec{w}) = \operatorname{div}\vec{v} + \operatorname{div}\vec{w}$$

$$\operatorname{div}\vec{r} = \operatorname{div}(x\vec{i} + y\vec{j} + z\vec{k}) = 1+1+1 = 3$$

b) $\vec{v} = (3x^2+y)\vec{i} - yz^3\vec{j} + z\cdot e^{2z-x}\vec{k}$

$\Longrightarrow P = 3x^2+y \quad \Longrightarrow P'_x = 6x$

$Q = -yz^3 \quad \Longrightarrow Q'_y = -z^3$

$R = z\cdot e^{2z-x} \quad \Longrightarrow R'_z = e^{2z-x} + z\cdot e^{2z-x}\cdot 2$

$$\Longrightarrow \operatorname{div}\vec{v} = 6x - z^3 + e^{2z-x}(1+2z).$$

4 a) Funcția Gamma $\Gamma(a) = \int_0^\infty e^{-x} \cdot x^{a-1} dx$, $a > 0$.

$$\Gamma(1) = 1$$
$$\Gamma(n) = (n-1)! \quad , \quad n \in \mathbb{N} \, , \; n \geq 2$$
$$\Gamma(a) = (a-1)\Gamma(a-1) \quad , \quad a > 1$$
$$\Gamma(a) \cdot \Gamma(1-a) = \frac{\pi}{\sin(\pi a)} \quad , \quad a \in (0,1).$$
$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

b) $\int_0^\infty x^2 \cdot e^{-x^2} dx$ Facem schimbarea de variabilă $x^2 = t$

$$x = 0 \Rightarrow t = 0^2 = 0$$
$$x = \infty \Rightarrow t = \infty^2 = \infty$$
$$x = \sqrt{t} = t^{\frac{1}{2}} \Rightarrow dx = \frac{1}{2} t^{\frac{1}{2}-1} dt$$

$$\Rightarrow \int_0^\infty x^2 \cdot e^{-x^2} dx = \int_0^\infty t \cdot e^{-t} \cdot \frac{1}{2} t^{\frac{1}{2}-1} dt = \frac{1}{2}\int_0^\infty e^{-t} \cdot t^{\frac{1}{2}} dt = \frac{1}{2}\int_0^\infty e^{-t} \cdot t^{1+\frac{1}{2}-1} dt$$
$$= \frac{1}{2}\Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right) = \frac{1}{4}\Gamma\left(\frac{1}{2}\right) = \frac{1}{4}\sqrt{\pi}.$$